

# MATRIX-BALL CONSTRUCTION OF AFFINE ROBINSON-SCHENSTED CORRESPONDENCE

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*In memory of V. A. Yasinskiy*

**ABSTRACT.** In his study of Kazhdan-Lusztig cells in affine type  $A$ , Shi has introduced an affine analog of Robinson-Schensted correspondence. We generalize the Matrix-Ball Construction of Viennot and Fulton to give a more combinatorial realization of Shi's algorithm. As a biproduct, we also give a way to realize the affine correspondence via the usual Robinson-Schensted bumping algorithm. Next, inspired by Lusztig and Xi, we extend the algorithm to a bijection between extended affine symmetric group and triples  $(P, Q, \rho)$  where  $P$  and  $Q$  are tabloids and  $\rho$  is a dominant weight. The weights  $\rho$  get a natural interpretation in terms of the Affine Matrix-Ball Construction. Finally, we prove that fibers of the inverse map possess a Weyl group symmetry, explaining the dominance condition on weights.

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## 1. INTRODUCTION

**1.1. Cells in Kazhdan-Lusztig theory.** In their groundbreaking paper [KL89] Kazhdan and Lusztig have laid a basis for an approach to representation theory of Hecke algebras. Since then this approach has been significantly developed, and is called *Kazhdan-Lusztig theory*. Of particular importance in it is the notion of *cells*.

Briefly, the definition is as follows. The Hecke algebra is associated with a Coxeter group  $W$ . Kazhdan and Lusztig define a pre-order on elements of  $W$  denoted  $\leq_L$ . Some pairs  $v, w$  of elements of  $W$  satisfy both  $v \leq_L w$  and  $w \leq_L v$ , in which case we say that they are left-equivalent, denoted  $v \sim_L w$ . Similarly one can define right equivalence  $\sim_R$ . The respective equivalence classes are called the *left cells* and the *right cells*. Another important notion turns out to be that of left-right, or two-sided equivalence  $\sim_{LR}$ . Two elements of  $W$  are left-right equivalent if they are connected by a series of left and right equivalences. The corresponding equivalence classes are called the *two-sided cells*.

**1.2. Type A.** In type A, i.e. when  $W$  is a symmetric group, the Kazhdan-Lusztig cell structure corresponds to something very familiar to combinatorialists, the *Robinson-Schensted correspondence*. It is a bijective correspondence between elements of a symmetric group and pairs of *standard Young tableaux* of the same shape. It is well known ([BV82], [KL89], [GM88], [Ari99]) that

- two permutations lie in the same left cell if and only if they have the same *recording tableau*  $\bar{Q}$ ;
- two permutations lie in the same right cell if and only if they have the same *insertion tableau*  $\bar{P}$ ;
- two permutations lie in the same two-sided cell if and only if their insertion and recording tableaux have the same shape  $\lambda$ ; equivalently, it happens when certain posets naturally associated with the permutations have the same *Greene-Kleitman invariants* [GK76].

We refer the reader to [Ful97] for details of the classical formulation of Robinson-Schensted insertion. Soon we will describe a less widely known construction, which can also be found in [Ful97], and which we generalize in this paper.

**1.3. Affine type A.** For  $W$  of affine type A, i.e. an affine symmetric group, Shi has shown in [Shi86] that the left Kazhdan-Lusztig cells correspond to *tabloids*. These are equivalence classes of fillings of Young diagrams with the first several integers (or residue classes) up to permuting elements within rows. The shape of these tabloids determines the two-sided cell. Furthermore, in [Shi91] Shi gave an algorithm for constructing a tabloid  $P(w)$  out of an affine permutation  $w \in W$ . We refer to this algorithm as *Shi's algorithm*. We will describe it in Section 9. We also refer to the affine Robinson-Schensted correspondence between left cells and tabloids as *Shi correspondence*.

Shi's algorithm is an involved process consisting of several distinct sub-algorithms which need to be applied in specific order. It makes it challenging to develop any direct intuitive connection between the permutation and the resulting tabloid. An alternative algorithm explained in Shi's paper (the one we will describe) is more natural in that it consists of a sequence of Knuth moves getting a permutation to a form where the tabloid can be read off. This version however requires pre-computing the Greene-Kleitman invariants of the associated poset.

The first goal of this paper is to find an alternative description of Shi correspondence. We look for an algorithm which would generalize in a natural way some known construction of the usual Robinson-Schensted correspondence, would yield insight into the meaning of associating a tabloid to a cell, and would be convenient in its applications to further study of Kazhdan-Lusztig cells in affine type  $A$ . It is our impression that the *Matrix-Ball Construction* of Viennot [Vie77] and Fulton [Ful97] is the most suitable candidate. Thus, in this paper we introduce the *Affine Matrix-Ball Construction*, or *AMBC*. The version of the construction that we generalize is essentially present in [Vie77], however we stay close to the further development of the ideas in [Ful97] in terms of terminology and exposition.

Surprisingly, as a biproduct of our analysis we get a realization of the Shi correspondence in terms of the usual Robinson-Schensted insertion, see Section 7 for details. A reader who wishes to learn a simple combinatorial way to realize the Shi correspondence can jump directly to this section.

**1.4. Bijectivity via weights.** A major difference between the finite and affine types is that in the latter case the map

$$\text{element } w \in W \mapsto (\text{insertion tabloid } P(w), \text{ recording tabloid } Q(w))$$

is *not* an injection. This is to be expected of course, as there are only finitely many pairs  $(P, Q)$ , while  $W$  is infinite.

One can get an actual bijection if one adds the third piece to the data: weights. Namely, there exists a bijection

$$\text{element } w \in W \mapsto (\text{insertion tabloid } P(w), \text{ recording tabloid } Q(w), \text{ dominant weight } \rho).$$

This was known since Lusztig's conjecture in [Lus89]. A proof and an explicit description of this correspondence was given by Xi [Xi02]. Later, a combinatorial formulation of Xi's construction was given by Honeywill [Hon05].

The second goal of this paper is to suggest a construction of such a bijection of our own. It turns out that weights  $\rho$  can be given a natural interpretation in terms of AMBC. We also explain how to complete the bijection by describing the inverse map  $(P, Q, \rho) \mapsto w$ . We leave the question of the exact relation between our weights and those of Xi for future research.

Throughout the paper we work with the extended affine symmetric groups, as does Honeywill in [Hon05]. In Section 10 we show that by imposing a simple restriction  $\sum_i \rho_i = 0$  on weights one recovers all of the theory for the non-extended case.

**1.5. Weyl group action.** We define the inverse map  $(P, Q, \rho) \mapsto w$  for any weight  $\rho$ , not just the dominant ones. This brings a natural question of describing its fibers: which  $\rho$ -s give the same  $w$ . It turns out that the fibers are orbits of a certain Weyl group acting on the lattice of weights. This explains why orbit representatives can be chosen to be dominant weights.

Proving this Weyl orbit property of the fibers is the third goal of this paper. We are unaware of anything similar to this theorem in the existing literature.

**1.6. Notational preliminaries.** Let  $n$  be a positive integer. Let  $[n] := \{1, \dots, n\}$ . For each  $i \in \mathbb{Z}$  denote by  $\bar{i}$  the residue class  $i + n\mathbb{Z}$ . Let  $\overline{[n]} := \{\bar{1}, \dots, \bar{n}\}$ .

We will encounter several groups. Let  $W$  be the symmetric group of type  $A_{n-1}$ . Let  $\widetilde{W}$  be the extended affine symmetric group of type  $\widetilde{A}_{n-1}$ ; it consists of elements  $w : \mathbb{Z} \rightarrow \mathbb{Z}$  such

that

$$w(i+n) = w(i) + n.$$

The elements of  $\widetilde{W}$  are called extended affine permutations. Since these will be common objects we deal with, we will sometimes shorten that to just permutations. Let  $\overline{W}$  be the affine symmetric group of type  $\tilde{A}_{n-1}$ ; it consists of elements  $w \in \widetilde{W}$  such that

$$\sum_{i=1}^n w(i) = \frac{n(n+1)}{2}.$$

A *partial (extended affine) permutation* is a pair  $(U, w)$  where  $U \subseteq \mathbb{Z}$  has the property that  $(x \in U) \Leftrightarrow (x+n \in U)$  and  $w : U \rightarrow \mathbb{Z}$  such that  $w(i+n) = w(i) + n$ . We will suppress the explicit mention of the subset  $U$  in the notation and just refer to the partial permutation  $w$ . Any permutation is viewed as a partial permutation with  $U = \mathbb{Z}$ .

A permutation is determined by its values on  $1, \dots, n$ . The *window notation* for a permutation  $w$  is  $[a_1, \dots, a_n]$  where  $w(1) = a_1, \dots, w(n) = a_n$ . A partial permutation  $w$  is also determined by its values on  $1, \dots, n$ , except it may not be defined on some of them. The *window notation* for a partial permutation  $w$  is  $[a_1, \dots, a_n]$  where  $a_i = w(i)$  if  $w$  is defined on  $i$  and  $a_i = \emptyset$  otherwise.

We usually think of permutations in terms of pictures such as the one in Figure 3. More precisely, on the plane we draw an infinite matrix; the rows are labeled by  $\mathbb{Z}$ , increasing downward, and the columns are labeled by  $\mathbb{Z}$ , increasing to the right. If  $w(i) = j$  then we place a ball in the  $i$ -th row and  $j$ -th column. To distinguish the 0-th row, we put a solid red line between the 0-th and 1-st rows, and similarly for columns. We also put dashed red lines every  $n$  rows and columns.

The cells of the matrix will be referred to by their matrix coordinates, e.g. the cell  $(1, 4)$  in Figure 3 contains a ball. The balls of a partial permutation will also be referred to by their matrix coordinates. For a partial permutation  $w$ , we denote by  $\mathcal{B}_w \subset \mathbb{Z} \times \mathbb{Z}$  the collection of balls of  $w$ .

For an integer  $k$  and a ball  $b = (i, j)$  we will refer to the ball  $b' = (i + kn, j + kn)$  as the  $k(n, n)$ -translate of  $b$  and denote this by  $b' = b + k(n, n)$ . Two balls  $b$  and  $b'$  are  $(n, n)$ -translates if for some  $k$  one is a  $k(n, n)$ -translate of the other.

We will assign numbers to balls of permutations as well as to other cells of the matrix. For a partial permutation  $w$ , a *numbering of  $w$*  is a function  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$ . A numbering  $d$  of  $w$  is *semi-periodic with period  $m$*  if for any  $b \in \mathcal{B}_w$  we have  $d(b + (n, n)) = d(b) + m$ . When referring to a numbering in pictures, we will write  $d(b)$  inside the ball  $b$  as done in Figure 3, where we show a semi-periodic numbering of period 3.

We use compass directions (e.g. north, east, northeast, etc.) inside the matrix with north being toward the top of the page and east being toward the right of the page. The relations are weak by default: a cell  $(i, j)$  is *southwest of*  $(i', j')$  if  $i \geq i'$  and  $j \leq j'$ . Adding the modifier “directly” constrains one of the two coordinates: a cell  $(i, j)$  is *directly south* of  $(i', j')$  if  $i \geq i'$  and  $j = j'$ . Directions define partial orders on  $\mathbb{Z} \times \mathbb{Z}$ : we say  $(i, j) \leq_{sw} (i', j')$  if  $(i, j)$  is southwest of  $(i', j')$ .

Sequences of cells going southeast and northwest will be particularly important to us. A *path* is a sequence  $(b_0, \dots, b_k)$  of cells such that for each  $i$ ,  $b_{i+1}$  is northwest of  $b_i$ . A *reverse path* is a sequence  $(b_0, \dots, b_k)$  of cells such that for each  $i$ ,  $b_{i+1}$  is southeast of  $b_i$ . In both cases, the number  $k$  is referred to as the *length* or *number of steps* of the (reverse) path.

## Part 1. Statements.

### 2. MATRIX-BALL CONSTRUCTION

In this section we briefly review the algorithm called the Matrix-Ball Construction (MBC) and its inverse. MBC is an implementation of the Robinson-Schensted correspondence which was originally described by Viennot ([Vie77]) in terms of shadows. It was later generalized by Fulton ([Ful97]) to the setting of integer matrices. We will be using the language from the second reference.

The Robinson-Schensted correspondence sends a non-affine permutation  $w \in W$  to a pair of standard Young tableaux  $(P(w), Q(w))$  of the same shape. Before describing the algorithm we will discuss a way to produce a partial non-affine permutation  $\text{fw}(w)$  from a partial non-affine permutation  $w$ . View  $w$  as an  $n \times n$  matrix with balls as described in the introduction. Make a numbering  $d_w$  of  $w$  iteratively as follows. For any ball  $b$  with no balls strictly northwest of it, let  $d_w(b) = 1$ . For any ball  $b$  with only balls numbered 1 strictly northwest of it, let  $d_w(b) = 2$ . Continue until all balls are numbered (see Figure 1 for examples of such numberings). For each  $i$ , form a Young diagram whose inner corners are the balls labeled  $i$  (the shape outlined by green and magenta lines in Figure 1). Note that sometimes there are no balls labeled  $i$ ; in this case we end up with the empty Young diagram which, by convention, has no outer corners, but has one inner corner. For each  $i$ , define  $(a_{w,i}, b_{w,i})$  to be the top-left cell of the  $i$ -th Young diagram; these are denoted by  $*$ 's in the figure. Let  $\text{fw}(w)$  be the partial non-affine permutation whose balls are the outer corners of the Young diagrams (the green balls in the figure). Now we can describe the algorithm.

- Input  $w \in W$ .
- Output: a pair  $(P(w), Q(w))$  of standard Young tableaux of the same shape.
- Initialize  $(P, Q)$  to  $(\emptyset, \emptyset)$ .
- Repeat until  $w$  is the empty partial permutation:
  - Add a row  $(b_{w,1}, b_{w,2}, \dots)$  to  $P$  and a row  $(a_{w,1}, a_{w,2}, \dots)$  to  $Q$ .
  - Reset  $w$  to  $\text{fw}(w)$ .
- Set  $P(w) = P$  and  $Q(w) = Q$ .

*Example 2.1.* Consider the non-affine permutation  $w = [5, 6, 1, 3, 4, 2]$ . The steps of AMBC are shown in Figure 1. The columns of  $*$ 's in the first picture are 1, 2, 4 so that is the first row of  $P(w)$ , while the rows are 1, 2, 5 so that is the first row of  $Q(w)$ . The remaining two rows are handled similarly, yielding

$$P(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & & \\ \hline \end{array} \quad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array}$$

It turns out that this procedure always produces a pair of standard Young tableaux of the same shape; moreover this is a bijection between  $W$  and the collection of pairs of Young tableaux of the same shape and of size  $n$ . Next we want to describe an algorithm which inverts the map  $w \mapsto (P(w), Q(w))$ . Just as with MBC, we start by describing an essential construction in the step.

Suppose  $w$  is a partial permutation and  $p = (p_1, \dots, p_k)$ ,  $q = (q_1, \dots, q_k)$  are two sequences of integers of the same length. The cells  $(p_i, q_i)$  are labeled by magenta numbers in Figure 2. View  $w$  as an  $n \times n$  matrix. Construct a numbering iteratively as follows. For a ball  $b$  with

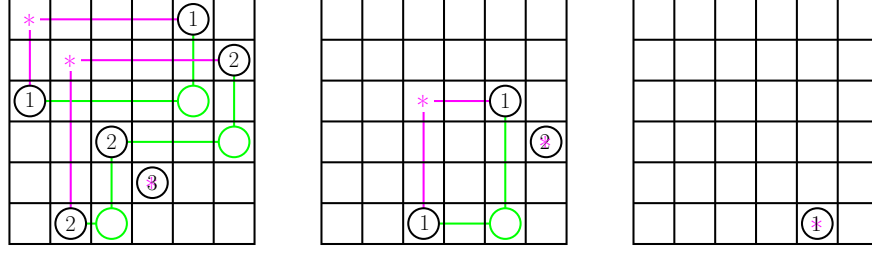


FIGURE 1. Steps of MBC for the non-affine permutation  $[5, 6, 1, 3, 4, 2]$ .

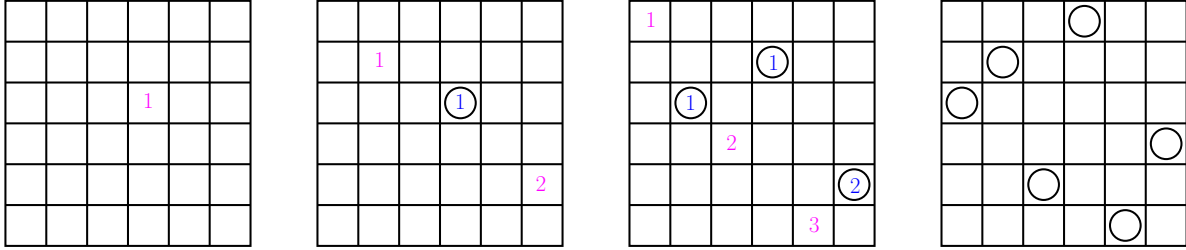


FIGURE 2. Steps of inverse MBC in Example 2.2.

no balls southeast of it, number  $b$  with the largest integer  $i$  such that  $(p_i, q_i)$  is northwest of  $b$ . Once all balls southeast of a given ball  $b$  have been numbered, number  $b$  with the largest integer  $i$  such that  $(p_i, q_i)$  is northwest of  $b$  and all balls southeast of  $b$  have numbers strictly larger than  $i$ . The resulting numbering is shown in blue in Figure 2. Now for each  $i$  form a Young diagram whose top-left corner is  $(p_i, q_i)$  and whose outer corners are the balls numbered  $i$ . Let  $\text{bk}_{p,q}(w)$  be the partial permutation whose balls are the inner corners of the Young diagram.

- Input a pair  $(P, Q)$  of standard Young tableaux of the same shape.
- Output:  $w \in W$ .
- Initialize  $w$  to the empty partial permutation.
- Repeat until  $(P, Q) = (\emptyset, \emptyset)$ :
  - Set  $p$  to be the last row of  $P$  and  $q$  to be the last row of  $Q$ . Remove the last rows from the tableaux.
  - Reset  $w$  to  $\text{bk}_{p,q}(w)$ .

*Example 2.2.* The steps of the algorithm for

$$P = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & & \\ \hline \end{array} \quad Q = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array}$$

are shown in Figure 2. The result is the non-affine permutation  $[4, 2, 1, 6, 3, 5]$ .

### 3. AFFINE MATRIX-BALL CONSTRUCTION

**3.1. Proper numberings.** In the step of MBC one began by numbering the balls of the permutation according to a certain rule. Attempting to directly apply the same rule to the

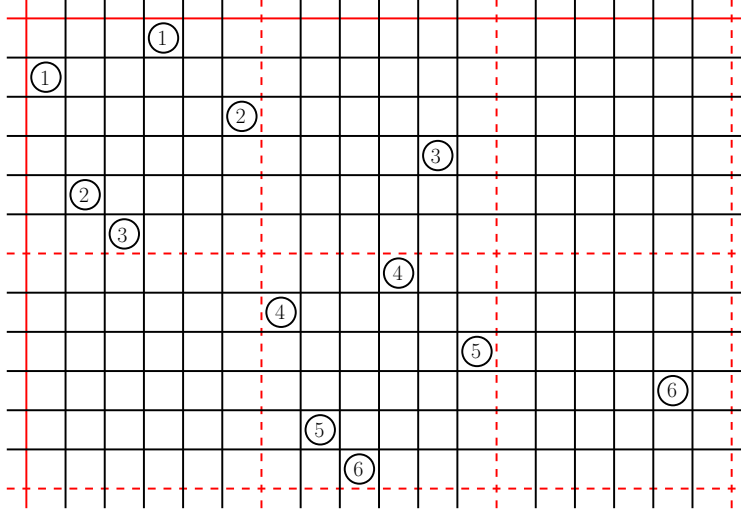


FIGURE 3. A proper numbering for the (extended) affine permutation  $[4, 1, 6, 11, 2, 3]$ . The numbering is semi-periodic with period 3.

balls of an affine permutation faces the problem that one does not know how to start (as will be described in Proposition 3.34, to some extent it does not matter). In this section we introduce a collection of numberings, called proper numberings, which work well to produce the two tabloids. In Section 3.3 we will choose a particular proper numbering which will be used in the forward step of the algorithm.

**Definition 3.1.** A function  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$  is a *proper numbering* if it is

- Monotone: for any  $b, b' \in \mathcal{B}_w$ , if  $b$  lies northwest of  $b'$  then  $d(b) < d(b')$ , and
- Continuous: for any  $b' \in \mathcal{B}_w$  there exists  $b$  northwest of it with  $d(b) = d(b') - 1$ .

An example of a proper numbering is given in Figure 3.

If we start with a proper numbering and increase the number of every ball by the same amount, then the resulting numbering is still proper. It is not apriori obvious that proper numberings exist. We will describe a construction of such a numbering in Section 3.3, and undertake a detailed study of their structure theory in Section 11.4.

**3.2. Shi poset.** Given an affine permutation  $w \in \widetilde{W}$ , Shi defined a (labeled) poset on  $[n]$ . As we shall see in Section 9, the Greene-Kleitman invariants (see [GK76] and a survey [BF01]) of this poset have significance in the Kazhdan-Lusztig theory of the affine symmetric group.

**Definition 3.2.** The (labeled) *Shi poset*  $P_w$  associated with  $w \in \widetilde{W}$  is the poset on  $[n]$  with  $i \leq_S j$  if

$$i > j, \text{ and } w(i) < w(j),$$

or if

$$w(j) > w(i) + n.$$

The element  $i$  is labeled by the residue class  $\overline{w(i)}$ .

*Example 3.3.* For  $w = [2, 8, 1, 14, 7, 16, 15, 0, 3, 9]$ , the Shi poset is shown in Figure 4.

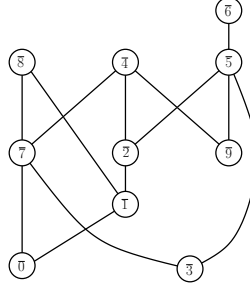


FIGURE 4. The Hasse diagram for the Shi poset associated with  $[2, 8, 1, 14, 7, 16, 15, 0, 3, 9]$ .

We can now state the first major result about the structure theory of proper numberings.

**Proposition 3.4.** *Any proper numbering is semi-periodic with period  $m$  equal to the width (i.e. maximal size of an antichain) of the Shi poset.*

The proof of this statement is found on page 29. The numbering in Figure 3 is semi-periodic with period 3.

**3.3. Channels and channel numberings.** In this section we introduce a collection of numberings which both provides the first examples of proper numberings and plays the most important role in AMBC.

**Definition 3.5.** The *projection map*  $\varphi_w : \mathcal{B}_w \rightarrow P_w$  is the map sending  $(i, w(i))$  to the representative of  $\bar{i}$  in  $[n]$ .

**Definition 3.6.** A *channel* is a preimage under the projection map of a longest antichain of the Shi poset.

The projections of two balls are incomparable in the Shi poset precisely when no two  $(n, n)$ -translates of the ball are comparable in the southwest ordering. Thus any channel can be described as follows. Consider a ball  $b$  and a reverse path  $B = (b_0 = b, b_1, b_2, \dots, b_k = b + (n, n))$ . If  $k$  is as large as possible (i.e.  $k = m$ ), then we can form a channel by taking all translates of  $B$ . Moreover, any channel can be obtained in this way.

Let  $\mathcal{C}_w$  be the set of all channels for a given  $w \in \tilde{A}_{n-1}$ . Fix a channel  $C \in \mathcal{C}_w$ . We will construct a proper numbering  $d_w^C$  as follows. Start with a proper numbering  $\tilde{d} : C \rightarrow \mathbb{Z}$  of  $C$  (there is an overall shift freedom). An example of such a numbering is shown in red in Figure 5.

**Definition 3.7.** Given a path  $(b_0, b_1, \dots, b_k)$  from a ball  $b_0$  to  $C$  (i.e.  $b_k \in C$ ), we will refer to the number  $\tilde{d}(b_k) + k$  as the *worth* of the path.

**Definition 3.8.** Suppose  $C$  is a channel and  $\tilde{d}$  is a proper numbering of it. Define the *channel numbering* of  $\mathcal{B}_w$  by

$$d_w^C(b) := \sup_{(b_0, b_1, \dots, b_k)} \tilde{d}(b_k) + k,$$

where the supremum is taken over all paths from  $b$  to  $C$ . Sometimes we will omit  $w$  from the notation when it is unambiguous and write just  $d^C$ .





The definition has an apparent asymmetry. However, it is easy to see that an equivalent definition would be:  $C_1$  is southwest of  $C_2$  if each element of  $C_2$  has at least one element of  $C_1$  southwest of it.

**Proposition 3.13.** *The southwest partial ordering on  $\mathcal{C}_w$  has a least element and a greatest element.*

The proof is found on page 26; in fact we prove that for any two channels, the set of southwest balls of their union is a channel and the set of northeast balls of their union is a channel. Thus there exists a southwest channel and a northeast channel. In AMBC we will be using the southwest channel numbering of balls.

Since the southwest channel numbering will play a key role in the paper, we introduce special notation for it.

**Definition 3.14.** Suppose  $w$  is a partial permutation. We write  $d_w^{SW}$  for the channel numbering with respect to the southwest channel.

Next we introduce a notion of distance between channels. To make sure that our definition is consistent we need a lemma:

**Lemma 3.15.** *Any proper numbering of  $w$  restricted to any channel  $C$  gives a proper numbering of the channel (i.e. a proper numbering labels elements of any channel by consecutive integers).*

The proof is found on page 32.

**Definition 3.16.** Let  $C_1, C_2 \in \mathcal{C}_w$  be two channels. Let  $a_1 : \mathcal{B}_w \rightarrow \mathbb{Z}$  be a channel numbering with respect to  $C_1$  (there is an overall shift freedom). Let  $a_2 : \mathcal{B}_w \rightarrow \mathbb{Z}$  be the channel numbering with respect to  $C_2$  which coincides with  $a_1$  on  $C_1$ . Define the *distance between  $C_1$  and  $C_2$*  by

$$h(C_1, C_2) = |a_2(b) - a_1(b)|,$$

where  $b$  is (any) ball of  $C_2$ .

**Proposition 3.17.** *For any partial permutation  $w$ , the function  $h$  defined above is a pseudometric on  $\mathcal{C}_w$ .*

The proof is found on page 29. As a pseudometric, this function naturally partition  $\mathcal{C}_w$  into equivalence classes:

**Definition 3.18.** A *river* is a maximal collection of channels such that the distance between any pair of them is 0.

*Example 3.19.* The channels of  $[4, 1, 11, 6, 14, 9, 3, 7, 17]$  are shown in Figure 6 (any channel has a blue curve going through it). This permutation has two rivers.

**3.4. Streams.** In this section we introduce the concept of streams. In MBC, the data necessary to invert a step of the algorithm consisted of two ordered subsets of  $[n]$  of the same size (the rows of the insertion and recording tableaux). The streams encode the additional data necessary to invert a step of AMBC.

Let  $A = \{\overline{a_1}, \dots, \overline{a_k}\}$  and  $B = \{\overline{b_1}, \dots, \overline{b_k}\}$  be two subsets of  $\overline{[n]}$  of the same size. Let  $\mathbf{st}(A, B)$  be a collection of subsets of elements of  $\mathbb{Z} \times \mathbb{Z}$  such that for any  $S \in \mathbf{st}(A, B)$

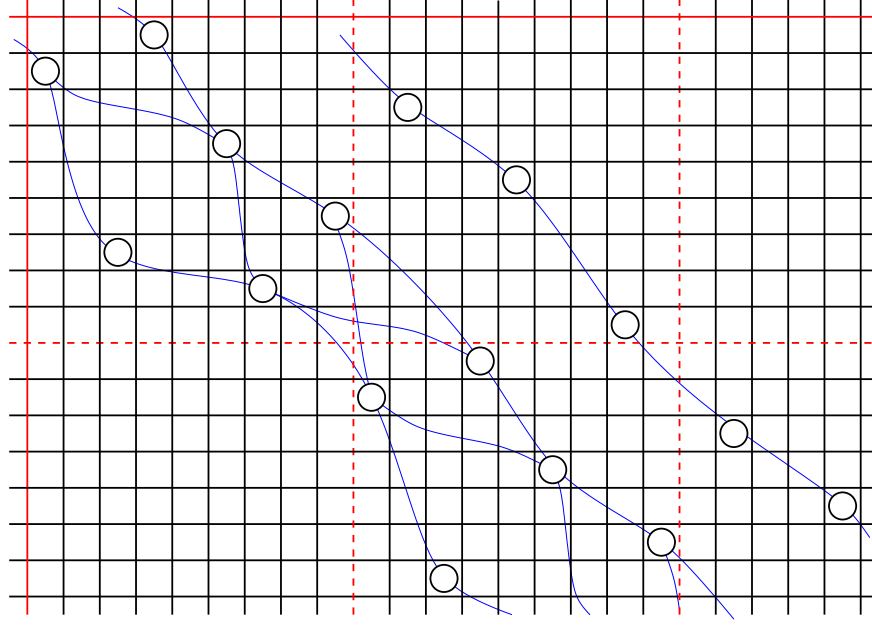


FIGURE 6. The channels of  $[4, 1, 11, 6, 14, 9, 3, 7, 17]$ . This permutation has two rivers.

- if  $(i, j) \in S$  then  $(i, j) \pm (n, n) \in S$ ;
- any two elements of  $S$  are comparable in SE order;
- $A = \{\bar{i} : (i, j) \in S\}$  and  $B = \{\bar{j} : (i, j) \in S\}$ .

**Definition 3.20.** We refer to elements of  $\mathfrak{st}(A, B)$  obtained in this way as *streams*, or  $(A, B)$ -streams when we want to specify the pair  $(A, B)$ .

**Definition 3.21.** The *flow* of an  $(A, B)$ -stream is the number  $|A| = |B|$ .

**Definition 3.22.** For a stream  $S$ , the *class* of  $S$  is the collection  $\mathfrak{st}(A, B)$  with  $S \in \mathfrak{st}(A, B)$ .

There are infinitely many  $(A, B)$ -streams, we classify them in the following way.

**Lemma 3.23.** *There is a unique stream  $\mathfrak{st}_0(A, B)$  such that there are exactly  $k = |A| = |B|$  of  $(i, j) \in \mathfrak{st}_0(A, B)$  such that  $1 \leq i, j \leq n$ .*

*Proof.* Let  $(a_1, \dots, a_k) \in [n]^k$  be the representatives of the classes in  $A$  in increasing order; similarly for  $B$ . It is clear that the unique  $(A, B)$ -stream satisfying the condition of the lemma is

$$\{(a_r + tn, b_r + tn) \mid 1 \leq r \leq k, t \in \mathbb{Z}\}.$$

□

Now, let

$$\mathfrak{st}_0(A, B) = \dots >_{SE} (i_{-1}, j_{-1}) >_{SE} (i_0, j_0) >_{SE} (i_1, j_1) >_{SE} \dots$$

be the list of elements of  $\mathfrak{st}_0(A, B)$  in the SE order.

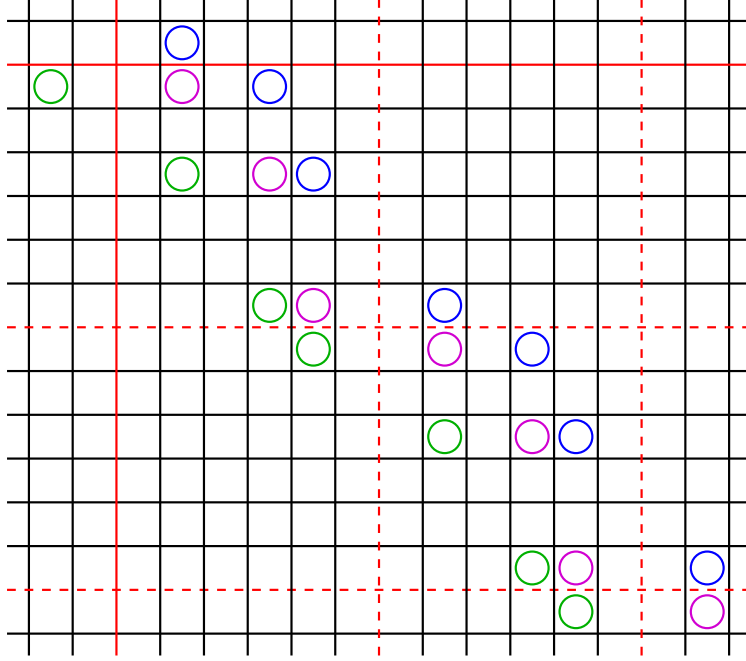


FIGURE 7. Streams of altitude  $-1, 0$ , and  $1$  for  $A = \{\bar{1}, \bar{3}, \bar{6}\}$  and  $B = \{\bar{2}, \bar{4}, \bar{5}\}$ .

**Proposition 3.24.** *For each  $r \in \mathbb{Z}$  let*

$$\mathfrak{st}_r(A, B) = \{(i_t, j_{t+r}) \mid t \in \mathbb{Z}\}.$$

*Then each  $\mathfrak{st}_r(A, B)$  is an  $(A, B)$ -stream, and all  $(A, B)$ -streams arise this way.*

*Proof.* Let  $S \in \mathfrak{st}(A, B)$ . Let  $(i_0, j) \in S$ , then clearly  $j = j_r$  for some  $r \in \mathbb{Z}$ . Then we claim  $(i_1, j_{1+r}) \in S$ . Indeed, if not, then the element of  $S$  in row  $i_1$  is weakly east of  $(i_1, j_{2+r})$ , which implies the element of  $S$  in row  $i_2$  is weakly east of  $(i_1, j_{3+r})$ , etc. This means however that  $S$  is missing an element in column  $j_{1+r}$ , which is impossible.  $\square$

**Definition 3.25.** The *altitude* of stream  $\mathfrak{st}_r(A, B)$  is the number  $r$ . For a stream  $S$ , we denote its altitude by  $a(S)$ .

*Example 3.26.* Let  $n = 6$ ,  $A = (1, 3, 6)$  and  $B = (2, 4, 5)$ . Then the streams  $\mathfrak{st}_{-1}(A, B)$ ,  $\mathfrak{st}_0(A, B)$  and  $\mathfrak{st}_1(A, B)$  are shown in Figure 7.

*Remark 3.27.* The data required to specify a stream is:

- a pair of subsets  $A, B$  of  $\overline{[n]}$  of the same size,
- an integer  $r$ .

**Definition 3.28.** The *defining data* of a stream  $S$  is the triple  $(A, B, r)$  such that  $S = \mathfrak{st}_r(A, B)$ .

**3.5. The map.** Let  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  be a partition of size  $\sum_i \lambda_i \leq n$ . A *tabloid*  $P$  of shape  $\lambda$  is an equivalence class of fillings of a Young diagram of shape  $\lambda$  with residues modulo  $n$  under identification of fillings that differ by reordering elements within rows. Let  $\Omega$  be the set of all triples  $(P, Q, \rho)$ , where  $P = (P_1, \dots, P_\ell)$  and  $Q = (Q_1, \dots, Q_\ell)$  are tabloids of

the same shape  $\lambda$  of size  $n$  filled with distinct residue classes, and  $\rho = (\rho_1, \dots, \rho_\ell) \in \mathbb{Z}^\ell$  is a *weight* of size  $\ell = \ell(\lambda)$ . Note that the  $i$ -th row in an element of  $\Omega$  describes a stream,  $\mathbf{st}_{\rho_i}(P_i, Q_i)$ .

**Definition 3.29.** Suppose  $w$  is a partial permutation and  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$  is a proper numbering. Define  $\text{fw}_d(w)$  to be the permutation obtained from  $w$  as done in the step of non-affine MBC. If  $C$  is a channel, we write  $\text{fw}_C(w)$  for  $\text{fw}_{d_C}(w)$ . If  $C$  is the southwest channel then we write  $\text{fw}(w)$  for  $\text{fw}_C(w)$ .

**Definition 3.30.** Suppose  $w$  is a partial permutation and  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$  is a proper numbering. Let  $b_r = (i_r, j_r)$ , where  $i_r$  is the row of the northern ball labeled  $r$  and  $j_r$  is the column of the western ball labeled  $r$ . Then  $\{b_r\}_{r \in \mathbb{Z}}$  forms a stream. We will refer to this stream as the *corner stream* of  $d$ . It will be denoted  $\mathbf{st}_d(w)$ . If  $C$  is a channel we will write  $\mathbf{st}_C(w)$  for  $\mathbf{st}_{d_C}(w)$ . We will write  $\mathbf{st}(w)$  for  $\mathbf{st}_{d_w^{\text{sw}}}(w)$ .

We are ready to define the map  $\Phi : \widetilde{W} \rightarrow \Omega$  which is the affine analog of Robinson-Schensted correspondence. We refer to the algorithm below as Affine Matrix-Ball Construction (AMBC).

- Input  $w \in \widetilde{W}$ .
- Output:  $(P, Q, \rho) \in \Omega$ .
- Initialize  $(P, Q, \rho)$  to  $(\emptyset, \emptyset, \emptyset)$ .
- Repeat until  $w$  is the empty partial permutation:
  - Record the defining data of  $\mathbf{st}(w)$ , denoted by \*'s in Figure 8, in the next row of  $P$ ,  $Q$ , and  $\rho$ .
  - Reset  $w$  to  $\text{fw}(w)$ .

**Theorem 3.31.** *This algorithm produces a valid element of  $\Omega$  at the end (i.e. the rows decrease in size and each tabloid is filled with distinct residue classes).*

The fact that each tabloid is filled with distinct residue classes is easy to see. The fact that rows decrease in size can be seen from, for example, the proof of Proposition 14.5.

*Example 3.32.* Consider  $w = [1, 2, 17, 5, 14, 18, 20]$  for  $n = 7$ . The steps of AMBC are shown in Figures 8 – 10. The resulting triple  $(P, Q, \rho)$  is

$$\begin{array}{|c|c|c|} \hline \overline{1} & \overline{2} & \overline{5} \\ \hline 0 & 4 & \overline{6} \\ \hline \overline{3} & & \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline \overline{0} & \overline{3} & \overline{6} \\ \hline \overline{2} & 4 & \overline{5} \\ \hline \overline{1} & & \\ \hline \end{array} \quad , \quad \begin{array}{|c|} \hline 3 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}$$

*Remark 3.33.* This map will be shown to be an injection, but not a surjection. In Section 5.2 we shall describe its image, while in Section 4 we shall describe a map whose restriction to the image is its inverse.

**Proposition 3.34.** *The two tabloids  $(P, Q)$  in the outcome do not change if we use an arbitrary proper numbering of balls at each step.*

The proof is found on page 36. From this result we see that the purpose of using the *southwest* channel numbering in AMBC is to get the desired weight  $\rho$ .

Motivated by the green parts of the pictures we introduce the following definition, which will be of great importance in the detailed analysis in the second half of the paper.

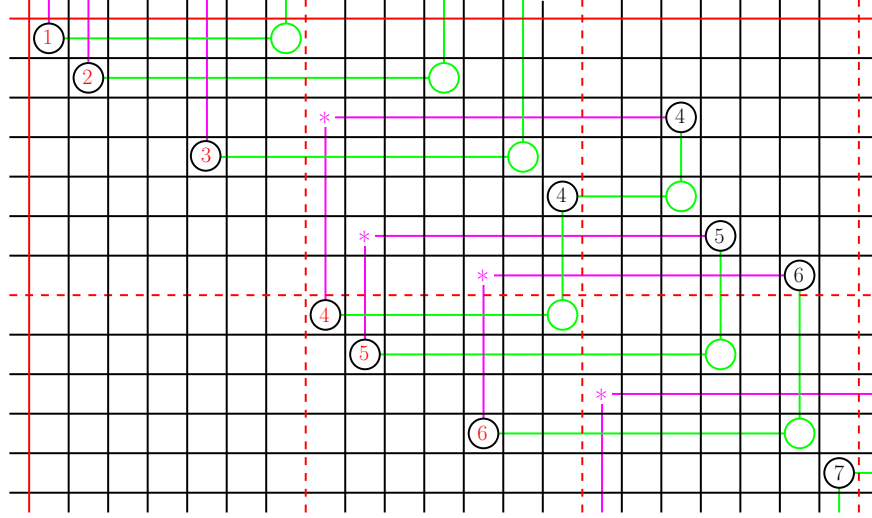


FIGURE 8. First step of AMBC for  $[1, 2, 17, 5, 14, 18, 20]$ . The numbering shown is the SW channel numbering; the balls in the channel are numbered in red. The new balls to be used in the next step are shown in green. The cells  $b_r$  in the algorithm are denoted by  $*$ 's. At the end of this step we set the first row of  $P$  to  $\{\bar{1}, \bar{2}, \bar{5}\}$ , the first row of  $Q$  to  $\{\bar{0}, \bar{3}, \bar{6}\}$ , and the first row of  $\rho$  to 3.

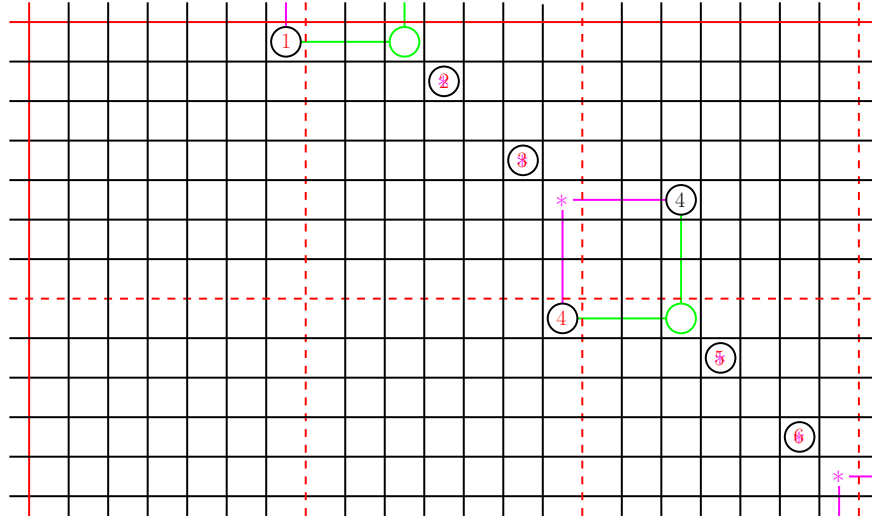


FIGURE 9. Second step of AMBC for  $[1, 2, 17, 5, 14, 18, 20]$ . At the end of this step we set the second row of  $P$  to  $\{\bar{0}, \bar{4}, \bar{6}\}$ , the second row of  $Q$  to  $\{\bar{2}, \bar{4}, \bar{5}\}$ , and the second row of  $\rho$  to 3.

**Definition 3.35.** A *zig-zag* is a finite sequence of cells  $((p_1, q_1), \dots, (p_k, q_k))$ , such that both of the following hold

- for all  $i$ , either  $(p_{i+1}, q_{i+1}) = (p_i - 1, q_i)$  or  $(p_{i+1}, q_{i+1}) = (p_i, q_i + 1)$  (informally, each step is either north or west),

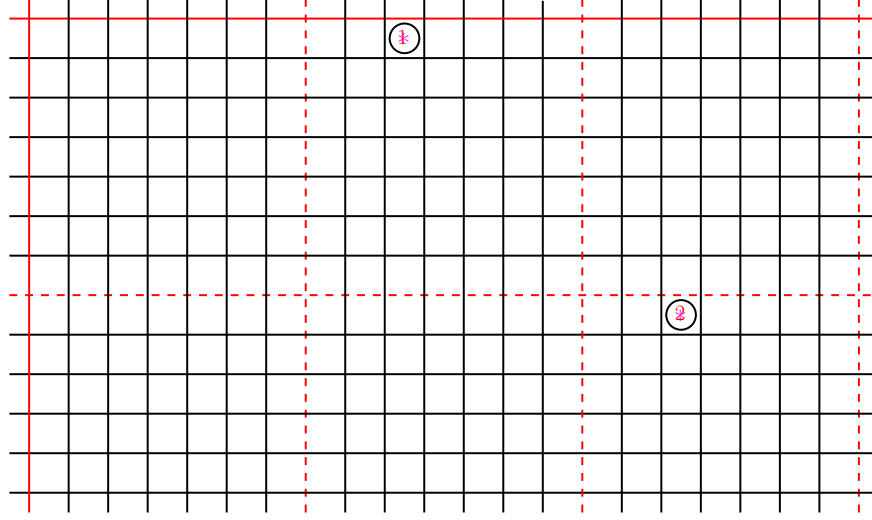


FIGURE 10. Third (and last) step of AMBC for  $[1, 2, 17, 5, 14, 18, 20]$ . At the end of this step we set the third row of  $P$  to  $\{\bar{3}\}$ , the third row of  $Q$  to  $\{\bar{1}\}$ , and the third row of  $\rho$  to 1.

- $(p_2, q_2) = (p_1 - 1, q_1)$  (the first step is north) and  $(p_k, q_k) = (p_{k-1}, q_{k-1} + 1)$  (the last step is west).

A *reverse zig-zag* is a finite sequence of cells  $((p_1, q_1), \dots, (p_k, q_k))$ , such that both of the following hold

- for all  $i$ , either  $(p_{i+1}, q_{i+1}) = (p_i - 1, q_i)$  or  $(p_{i+1}, q_{i+1}) = (p_i, q_i + 1)$  (informally, each step is either north or west),
- $(p_2, q_2) = (p_1, q_1 + 1)$  (the first step is west) and  $(p_k, q_k) = (p_{k-1} - 1, q_{k-1})$  (the last step is north).

The reverse designation is motivated by the fact that in Section 14 we will use zig-zags to create paths and reverse zig-zags to create reverse paths. It is clear that any proper numbering of a partial permutation  $w$  defines a collection of (numbered) reverse zig-zags.

#### 4. BACKWARD ALGORITHM

We have introduced AMBC which gives a map  $\Phi : \widetilde{W} \rightarrow \Omega$ . In this section we will introduce an algorithm, which we will call the backward algorithm, that gives a map  $\Psi : \Omega \rightarrow \widetilde{W}$ . The restriction of  $\Psi$  to the image of  $\Phi$  will be precisely the inverse of  $\Phi$ . The behavior of  $\Psi$  on the rest of  $\Omega$  will be explained in Section 6 (including a description of the fibers  $\Psi^{-1}(w)$ ); essentially applying the backward algorithm followed by the forward algorithm preserves the two tabloids and maps the weight into a certain (shifted) dominant chamber.

**4.1. Backward numberings.** First we explain a way to construct a numbering of the balls of a partial permutation with respect to a stream. This numbering will mimic the numbering of balls in the inverse MBC algorithm.

**Definition 4.1.** We say that a stream  $S$  is *compatible* with a partial permutation  $w$  if

- $\mathcal{B}_w \cup S$  has at most one cell in each row and at most one cell in each column,
- the flow of  $S$  is greater than or equal to the width of  $P_w$ .

The numbering will be defined by an algorithm:

- Input: a partial affine permutation  $w$  and a compatible stream  $S$ , and a proper numbering  $f : S \rightarrow \mathbb{Z}$ .
- Output: a numbering  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$ .
- Construct  $d$  inductively. For  $b \in \mathcal{B}_w$ , let  $d_0(b)$  be the largest value that  $f$  takes northwest of  $b$ . This gives a numbering  $d_0 : \mathcal{B}_w \rightarrow \mathbb{Z}$ .
- Assume  $d_i$  is defined. If  $d_i$  is increasing in the sense that for any  $b$  strictly northwest of  $b'$  we have  $d_i(b) < d_i(b')$ , let  $d = d_i$  and the algorithm terminates. Otherwise choose a ball  $b$  such that there are balls southeast of it with the same value of  $d_i$ , but no such balls northwest of it. Define  $d_{i+1}$  to be the semi-periodic numbering which coincides with  $d_i$  on all balls which are not  $(n, n)$ -translates of  $b$  and has  $d_{i+1}(b) = d_i(b) - 1$ .

**Definition 4.2.** We refer to the numbering  $d_0$  constructed in the above algorithm as the *stream numbering* of  $w$ .

**Proposition 4.3.** *The above algorithm terminates on any valid input and the numbering  $d$  does not depend on the choices made. We refer to this common numbering as the backward numbering corresponding to a partial permutation and a stream.*

The proof is found on page 40.

*Example 4.4.* Consider the partial permutation  $[6, 1, \emptyset, 8, 5, 7, \emptyset, \emptyset] \in \tilde{A}_7$  and the stream  $\mathbf{st}_0(\{\bar{3}, \bar{7}, \bar{8}\}, \{\bar{2}, \bar{3}, \bar{4}\})$ . Number  $S$  so that the cell labeled 1 is the northwest cell both of whose coordinates are positive. The process of obtaining the corresponding backward numbering is illustrated in Figure 11.

Finally, we introduce additional notation.

**Definition 4.5.** Suppose  $S$  is a stream and  $f$  is a proper numbering of it. Then by  $S^{(i)}$  we mean the cell of  $S$  numbered  $i$ .

Often when dealing with the backward algorithm, the proper numbering  $f$  is implicit since the overall shift is of no interest. We will only need to talk about it explicitly when we compare the results of the backward step with different streams and hence care about their relative numberings.

**4.2. The map.** Now we are ready to describe the backward algorithm (map  $\Psi$ ):

- Input:  $(P, Q, \rho) \in \Omega$ .
- Output:  $w \in \tilde{W}$ .
- Initialize  $w$  to the empty partial permutation.
- Repeat until  $(P, Q, \rho) = (\emptyset, \emptyset, \emptyset)$ :
  - Remove the last row of  $P$  and call it  $A$ ; remove the last row of  $Q$  and call it  $B$ ; remove the last row of  $\rho$  and call it  $r$ .
  - Let  $S = \mathbf{st}_r(A, B)$  be the stream corresponding to the data; give  $S$  some proper numbering.
  - Let  $d : \mathcal{B}_w \rightarrow \mathbb{Z}$  be the backward numbering corresponding to  $S$ .



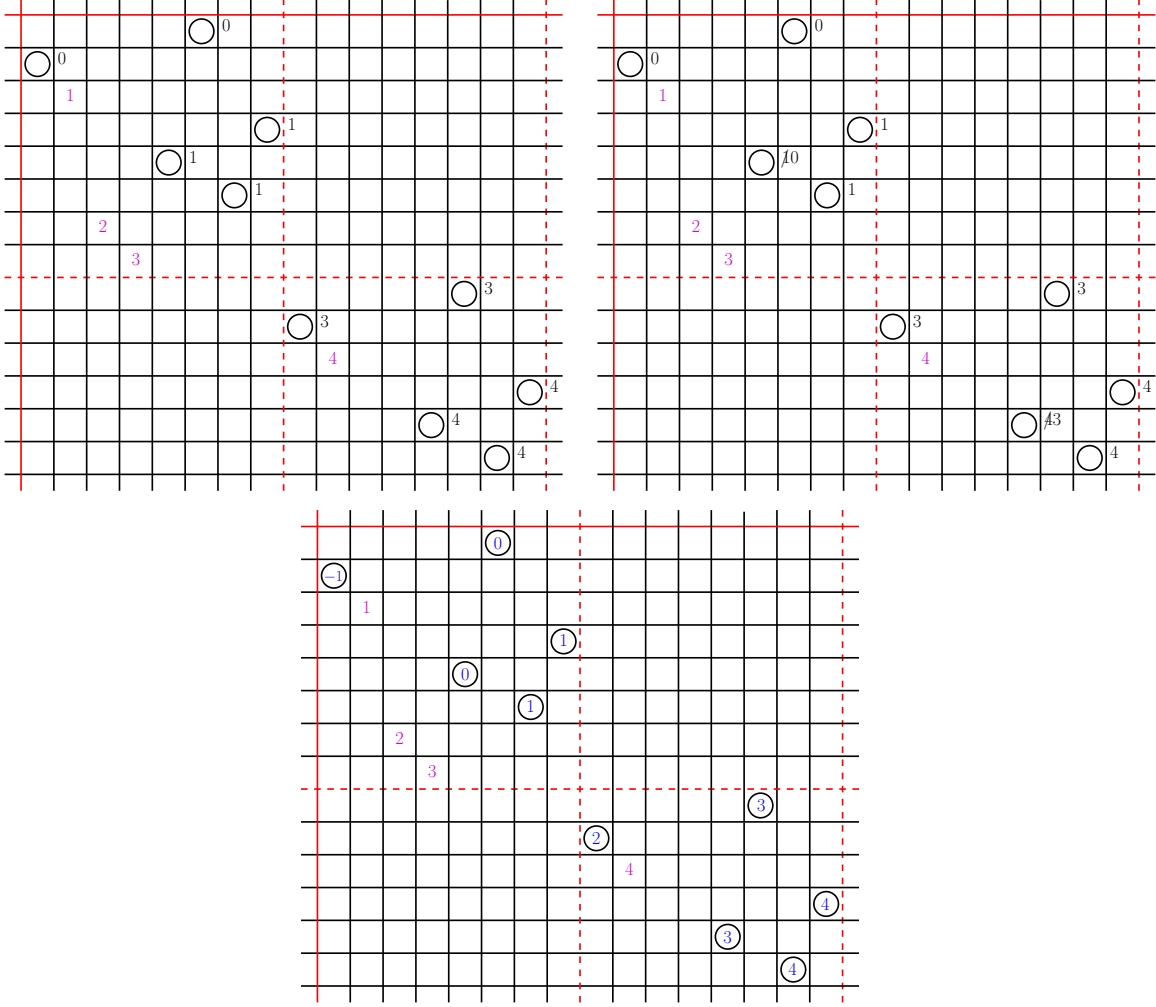


FIGURE 11. The process of obtaining the backward numbering of  $[6, 1, \emptyset, 8, 5, 7, \emptyset, \emptyset]$  with respect to the stream  $\mathfrak{st}_0(\{\bar{3}, \bar{7}, \bar{8}\}, \{\bar{2}, \bar{3}, \bar{4}\})$ . The cells of the stream have magenta numbers; these numbers correspond to the values of  $f$  on the cells. The top-left figure shows the initial numbering  $d_0$ , the top-right figure shows  $d_1$ , and the bottom figure shows the backward numbering  $d = d_2$ .

- For each  $i \in \mathbb{Z}$  form a Young diagram with top-left corner at the cell of  $S$  labeled  $i$  and outer corners at the balls of  $\mathcal{B}_w$  labeled  $i$  (see Figure 12). Note that sometimes there are no balls labeled  $i$ ; in this case we end up with the empty Young diagram which, by convention, has no outer corners, but has one inner corner.
- Consider the partial permutation  $\text{bk}_S(w)$  consisting of the inner corners of the Young diagram.
- Let  $w = \text{bk}_S(w)$ .
- Output  $w$ .

*Example 4.6.* We will apply the backward algorithm to the following element of  $\Omega$ :

$$(P, Q, \rho) = \left( \begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{3} & \bar{5} & \bar{7} \\ \hline \bar{1} & \bar{4} & & \\ \hline \bar{6} & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{4} & \bar{5} & \bar{7} \\ \hline \bar{3} & \bar{6} & & \\ \hline \bar{2} & & & \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 0 \\ \hline 1 \\ \hline \end{array} \right).$$

Taking off the last row gives us a stream  $S_1$  of flow 1. Since the partial permutation at the beginning of the step is empty, all the young diagrams will be empty. Thus the partial permutation at the end of the step,  $w_1 = \text{bk}_{S_1}(\emptyset)$  will simply consist of the cells of  $S_1$ . This is shown in the top-left part of Figure 12. We are left with

$$\left( \begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{3} & \bar{5} & \bar{7} \\ \hline \bar{1} & \bar{4} & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{4} & \bar{5} & \bar{7} \\ \hline \bar{3} & \bar{6} & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 0 \\ \hline \end{array} \right).$$

Taking off the last row gives the stream  $S_2$  shown by magenta numbers in the top-right part of Figure 12. The black balls show the partial permutation  $w_1$ ; the numbering is the backward numbering with respect to  $S_2$ . The blue balls show the new partial permutation  $w_2 = \text{bk}_{S_2}(w_1)$ . Finally, the bottom of Figure 12 shows the last step of the backward algorithm; the blue balls give the permutation

$$\Psi(P, Q, \rho) = [1, 6, 9, 7, 10, 5, 11] \in \tilde{A}_6.$$

## 5. BIJECTIVITY

In this section we briefly discuss injectivity of  $\Phi$  and then describe the image of  $\Phi$  as a subset of  $\Omega$ . This description will have the form: a collection of triples  $(P, Q, \rho) \in \Omega$ , where  $\rho$  satisfies certain inequalities which depend on  $P$  and  $Q$ . The primary goal is to describe the inequalities explicitly.

The first result, which implies injectivity, is that AMBC post-composed with the backward algorithm gives the identity.

**Theorem 5.1.** *Suppose  $w$  is a partial permutation. Then  $\Psi(\Phi(w)) = w$ .*

This is actually true at each step regardless of which channel numbering we use:

**Proposition 5.2.** *Let  $w$  be a partial permutation and let  $C$  be a channel. Then*

$$\text{bk}_{\text{st}_C(w)}(\text{fw}_C(w)) = w.$$

The proof is found on page 51. Notice that the proposition is valid only for channel numberings, not for general proper numberings.

### 5.1. Concurrent streams.

**Lemma 5.3.** *Let  $S, T$  be two streams of the same flow. Let  $t$  be the partial permutation whose balls are the cells of  $T$ ; assume  $S$  is compatible with  $t$ . Then  $\text{bk}_S(t)$  can be partitioned into two disjoint channels.*

The proof is found on page 44.

**Definition 5.4.** Let  $S, T$  be two streams of the same flow. Let  $t$  be the partial permutation whose balls are the cells of  $T$ ; assume  $S$  is compatible with  $t$ . Then  $T$  is *concurrent to  $S$*  if the two disjoint channels of  $\text{bk}_S(t)$  are part of the same river (i.e. the distance between them is zero).

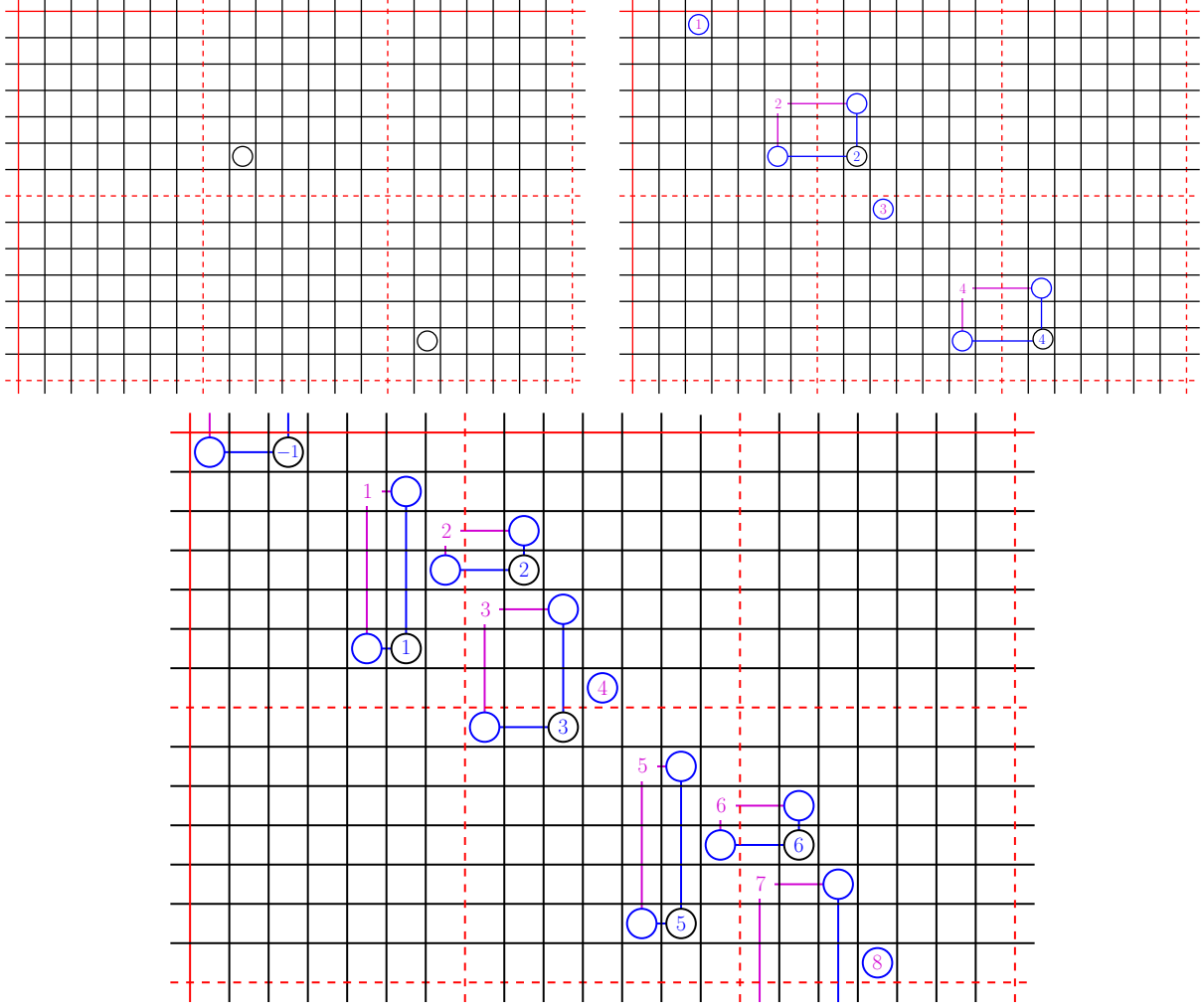


FIGURE 12. The steps in the application of the backward algorithm described in Example 4.6.

*Remark 5.5.* Note that the definition is not symmetric;  $T := \mathfrak{st}_0(\{\bar{3}, \bar{4}, \bar{5}\}, \{\bar{1}, \bar{3}, \bar{5}\})$  is concurrent to  $S := \mathfrak{st}_0(\{\bar{1}, \bar{2}, \bar{6}\}, \{\bar{2}, \bar{4}, \bar{6}\})$ , but  $S$  is not concurrent to  $T$ . This suggests that the terminology may be suboptimal.

**Proposition 5.6.** *Consider four sets  $A, A', B, B' \subset [n]$ , all of the same size, with  $A$  disjoint from  $A'$  and  $B$  disjoint from  $B'$ . Then there exists a unique  $r \in \mathbb{Z}$  such that  $\mathfrak{st}_r(A', B')$  is concurrent to  $\mathfrak{st}_0(A, B)$ . Moreover, for any  $l$ ,  $\mathfrak{st}_{r+l}(A', B')$  is concurrent to  $\mathfrak{st}_l(A, B)$ .*

The proof is found on page 44.

*Example 5.7.* Suppose  $n = 6$ ,  $S = \mathfrak{st}_0(\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{2}, \bar{6}\})$ . Then the stream from the class  $\mathfrak{st}(\{\bar{2}, \bar{4}, \bar{6}\}, \{\bar{3}, \bar{4}, \bar{5}\})$  which is concurrent to  $S$  is  $\mathfrak{st}_1(\{\bar{2}, \bar{4}, \bar{6}\}, \{\bar{3}, \bar{4}, \bar{5}\})$ . The two streams are shown in Figure 13.

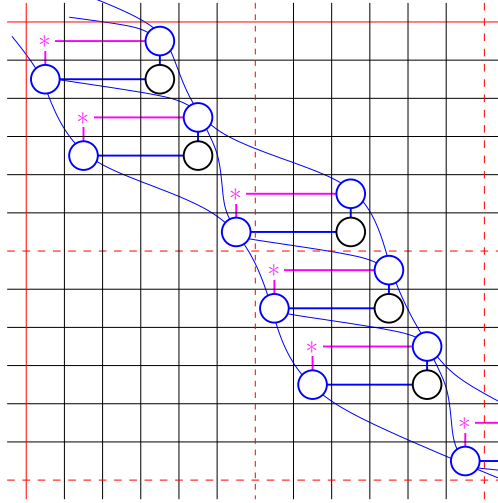


FIGURE 13. The stream  $\mathbf{st}_1(\{\bar{2}, \bar{4}, \bar{6}\}, \{\bar{3}, \bar{4}, \bar{5}\})$ , denoted by black balls, is concurrent to  $\mathbf{st}_0(\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{2}, \bar{6}\})$ , denoted by magenta  $*$ 's, since the backward step results in one river.

**5.2. Dominant weights.** Suppose  $(P, Q, \rho) \in \Omega$  and  $P$  and  $Q$  have shape  $\lambda$ . Let  $r_1 = 0$ . For each  $2 \leq i \leq \ell(\lambda)$ : if  $\lambda_i < \lambda_{i-1}$  then let  $r_i = 0$ , otherwise let  $r_i$  be the unique integer such that  $\mathbf{st}_{r_i}(P_i, Q_i)$  is concurrent to  $\mathbf{st}_0(P_{i-1}, Q_{i-1})$ .

**Definition 5.8.** Consider  $(P, Q, \rho) \in \Omega$ . Suppose  $P$  and  $Q$  have shape  $\lambda$ . Then  $\rho$  is *dominant* if for each  $1 \leq i < l(P)$  one of the following holds:

- $\lambda_i > \lambda_{i+1}$ , or
- $\lambda_i = \lambda_{i+1}$  and  $\rho_{i+1} - r_{i+1} \geq \rho_i - r_i$ .

We introduce special notation for the dominant part of  $\Omega$ .

**Definition 5.9.** Let  $\Omega_{dom} := \{(P, Q, \rho) \in \Omega : \rho \text{ is dominant}\}$

**Theorem 5.10.**  $\Phi(\widetilde{W}) \subseteq \Omega_{dom}$ .

The proof is found on page 50.

**Theorem 5.11.** Suppose  $(P, Q, \rho) \in \Omega_{dom}$ . Then  $\Phi(\Psi(P, Q, \rho)) = (P, Q, \rho)$ .

The proof is found on page 54.

*Remark 5.12.* In the standard presentation of the Lie algebra  $\mathfrak{sl}_n$ , the dominant integral weights are increasing sequences of integers. Thus our dominant weights are actually located in some shift of the dominant chamber of a product of special linear Lie algebras.

## 6. WEYL GROUP ACTION

Theorem 5.11 describes the composition  $\Phi \circ \Psi$  when the weight is dominant. In this section we describe what happens if we started with a non-dominant weight.

**Definition 6.1.** Let  $(P, Q, \rho) \in \Omega$  with  $\text{sh}(P) = \text{sh}(Q) = \lambda$  and for  $1 \leq i \leq \ell(\lambda)$  define  $r_i$  as in Section 5.2 and let  $\mathbf{r} = (r_i)_i$ . Let  $i_1$  be the smallest index for which  $\lambda_{i_1} < \lambda_1$ ,  $i_2$  be

the smallest index for which  $\lambda_{i_2} < \lambda_{i_1}$ , etc.; suppose  $i_l$  is the last one of these. Define the *dominant representative* of  $\rho$  to be the weight  $\rho'$  obtained as follows:

- Let  $\mathbf{x} = \rho - \mathbf{r}$ .
- For each  $1 \leq j < l$ , let  $(y_{i_j}, y_{i_j+1}, \dots, y_{i_{j+1}-1})$  be the increasing rearrangement of  $(x_{i_j}, x_{i_j+1}, \dots, x_{i_{j+1}-1})$ .
- Let  $\rho' = \mathbf{y} + \mathbf{r}$ .

Thus the dominant representative of  $\rho$  is precisely what one would get by mapping the weight  $\rho$  into the (shifted) dominant chamber by reflections in the (shifted) hyperplanes defining it.

*Example 6.2.* Consider the following element of  $\Omega$ :

$$(P, Q, \rho) = \left( \begin{array}{|c|c|c|} \hline \bar{1} & \bar{2} & \bar{3} \\ \hline \bar{5} & \bar{7} & \bar{10} \\ \hline \bar{4} & \bar{8} & \bar{11} \\ \hline \bar{12} & \bar{13} & \\ \hline \bar{6} & \bar{9} & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bar{1} & \bar{2} & \bar{5} \\ \hline \bar{3} & \bar{8} & \bar{12} \\ \hline \bar{4} & \bar{10} & \bar{11} \\ \hline \bar{6} & \bar{13} & \\ \hline \bar{7} & \bar{9} & \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \right).$$

The vector  $r$  in this case is  $r = (r_i) = (0, 0, 0, 0, 1)$ . So the vector  $\rho - r$  is  $(2, 1, 0, 0, -1)$ . Now reshuffling the first three entries to be in increasing order (since the first three rows of the tabloids have the same size), and the last two in increasing order gives:  $(0, 1, 2, -1, 0)$ . To get  $\rho'$  we add  $r$ :  $\rho' = (0, 1, 2, -1, 1)$ .

**Theorem 6.3.** *Let  $(P, Q, \rho) \in \Omega$ . Then  $\Phi(\Psi(P, Q, \rho)) = (P, Q, \rho')$ , where  $\rho'$  is the dominant representative of  $\rho$ .*

The proof is found on page 70. As an easy corollary we can describe the fibers  $\Psi^{-1}(w)$ .

**Corollary 6.4.** *Suppose  $P$  and  $Q$  are two tabloids with the same shape  $\lambda$ ; let  $r = (r_1, \dots, r_{\ell(\lambda)})$  be the vector of shifts described in Section 5.2. Let  $S$  be the parabolic subgroup of the permutation group  $S_{\ell(\lambda)}$  generated by transpositions  $s_i$  such that  $\lambda_i = \lambda_{i+1}$ . Then  $\Psi(P, Q, \rho) = \Psi(P, Q, \rho')$  if and only if  $\rho - \mathbf{r} = s(\rho' - \mathbf{r})$  for some  $s \in S$ .*

## 7. ASYMPTOTIC REALIZATION VIA THE USUAL ROBINSON-SCHENSTED INSERTION

In this section we describe how AMBC is related to the bumping algorithm. Any  $w \in \tilde{A}_{n-1}$  can be presented as an infinite semi-periodic sequence of integers:

$$\dots, w(-1), w(0), w(1), w(2), \dots,$$

where  $w(i+n) = w(i) + n$ . One can then apply the usual RSK algorithm to insert this sequence, choosing the initial place arbitrarily, for example at  $w(1)$ .

*Example 7.1.* Let  $n = 6$  and let  $w = [-4, 5, -2, 7, 3, 6]$ . Then the sequence of  $w(i)$ ,  $i \geq 1$  looks as follows:

$$-4, 5, -2, 7, 3, 6, 2, 11, 4, 13, 9, 12, 8, \dots$$

Inserting this sequence, we obtain a sequence of tableaux

$$\begin{array}{|c|} \hline -4 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline -4 & 5 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline -4 & -2 \\ \hline 5 & \end{array}, \quad \begin{array}{|c|c|c|} \hline -4 & -2 & 7 \\ \hline 5 & & \end{array}, \quad \begin{array}{|c|c|c|} \hline -4 & -2 & 3 \\ \hline 5 & 7 & \end{array}, \quad \begin{array}{|c|c|c|c|} \hline -4 & -2 & 3 & 6 \\ \hline 5 & 7 & & \end{array},$$

$$\begin{array}{|c|c|c|c|} \hline -4 & -2 & 2 & 6 \\ \hline 3 & 7 & & \\ \hline 5 & & & \\ \hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|} \hline -4 & -2 & 2 & 6 & 11 \\ \hline 3 & 7 & & & \\ \hline 5 & & & & \\ \hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|} \hline -4 & -2 & 2 & 4 & 11 \\ \hline 3 & 6 & & & \\ \hline 5 & 7 & & & \\ \hline \end{array}, \dots$$

Denote  $\bar{P}_i(w)$  the tableau obtained by inserting  $w(1), \dots, w(i)$ . One can then create the associated tabloid  $P_i(w)$  by passing from each number to its residue modulo  $n$  and then forgetting the order of elements in each row.

*Example 7.2.* The above sequence of tableaux results in the following sequence of tabloids.

$$\begin{array}{|c|}, \begin{array}{|c|c|} \hline \bar{2} & \bar{5} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \bar{2} & \bar{4} \\ \hline \bar{5} & \end{array}, \begin{array}{|c|c|c|} \hline \bar{1} & \bar{2} & \bar{4} \\ \hline \bar{5} & & \end{array}, \begin{array}{|c|c|c|} \hline \bar{2} & \bar{3} & \bar{4} \\ \hline \bar{1} & \bar{5} & \end{array}, \begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{3} & \bar{4} & \bar{6} \\ \hline \bar{1} & \bar{5} & & \end{array}, \begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{2} & \bar{4} & \bar{6} \\ \hline \bar{1} & \bar{3} & & \end{array}, \begin{array}{|c|c|c|c|c|} \hline \bar{2} & \bar{2} & \bar{4} & \bar{5} & \bar{6} \\ \hline \bar{1} & \bar{3} & & & \end{array}, \begin{array}{|c|c|c|c|c|} \hline \bar{2} & \bar{2} & \bar{4} & \bar{4} & \bar{5} \\ \hline \bar{3} & \bar{6} & & & \end{array}, \dots$$

**Theorem 7.3.** For large enough  $i$  we have

$$P_{i+n}(w) = P_i(w) + P(w)$$

where  $P(w)$  is the tabloid obtained by Shi insertion of  $w$ , and content of tabloids is added as union of multisets: first row to first row, second row to second row, etc.

The proof is found on page 36.

*Example 7.4.* In the example above we have

$$\bar{P}_{13}(w) = \begin{array}{|c|c|c|c|c|c|} \hline -4 & -2 & 2 & 4 & 8 & 12 \\ \hline 3 & 6 & 9 & 13 & & \\ \hline 5 & 7 & 11 & & & \\ \hline \end{array},$$

which gives

$$P_{13}(w) = \begin{array}{|c|c|c|c|c|c|} \hline \bar{2} & \bar{2} & \bar{2} & \bar{4} & \bar{4} & \bar{6} \\ \hline \bar{1} & \bar{3} & \bar{3} & \bar{6} & & \\ \hline \bar{1} & \bar{5} & \bar{5} & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{2} & \bar{4} & \bar{6} \\ \hline \bar{1} & \bar{3} & & \\ \hline \bar{5} & & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bar{2} & \bar{4} \\ \hline \bar{3} & \bar{6} \\ \hline \bar{1} & \bar{5} \\ \hline \end{array} = P_7(w) + P(w).$$

This allows one to obtain the following asymptotic version of Shi's insertion.

**Corollary 7.5.** The number of  $j$ 's in the  $k$ -th row of  $P(w)$  (which may only be 0 or 1) is equal to

$$\lim_{i \rightarrow \infty} \frac{\text{number of } j\text{'s in } k\text{-th row of } P_i(w)}{i/n}.$$

*Remark 7.6.* Note that this can be used to compute the Shi insertion  $P(w)$  in practice. Indeed, once  $i$  is sufficiently large,  $P_i(w)$  becomes a multiple of  $P(w)$  plus small “noise”. One can easily tell this noise apart, finding the  $P(w)$ . For example, in the example above we have

$$P_{20}(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 2 & 4 & 4 & 4 & 5 & 6 \\ \hline 1 & 3 & 3 & 3 & 6 & 6 & & & \\ \hline 1 & 1 & 5 & 5 & 5 & & & & \\ \hline \end{array},$$

from which one can gauge that

$$P(w) = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 6 \\ \hline 1 & 5 \\ \hline \end{array}.$$

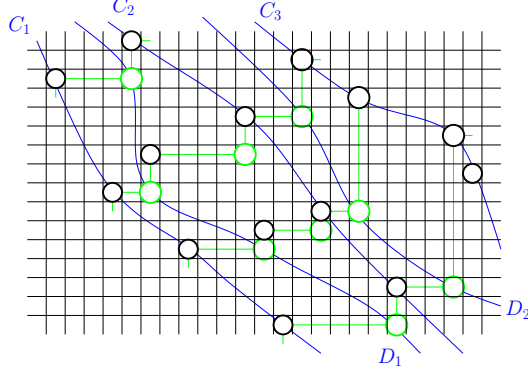


FIGURE 14. The collection  $\{D_1, D_2\}$  of channels of  $\text{fw}(w)$  interlaces the collection  $\{C_1, C_2, C_3\}$  of channels of  $w$ .

*Remark 7.7.* One can also get the tabloid  $Q(w)$  via the limits of the usual Robinson-Schensted insertion by inserting  $w^{-1}$ .

## 8. COMBINATORIAL INTERPRETATION OF WEIGHTS

Suppose  $w$  is a partial permutation which has  $k \geq 2$  disjoint channels. As mentioned with regard to Proposition 3.13, the collection of southwest balls of a union of two channels is a channel. Thus we can organize a maximal disjoint collection  $\{C_1, C_2, \dots, C_k\}$  of channels of  $w$  such that for all  $i$ ,  $C_i$  is southwest of  $C_{i+1}$ . As we will see in Section 14.2, there exists a maximal disjoint collection  $\{D_1, D_2, \dots, D_{k-1}\}$  of channels of  $\text{fw}(w)$  which interlace  $\{C_1, C_2, \dots, C_k\}$  as shown in Figure 14.

**Theorem 8.1.** *Suppose  $w$  is a partial permutation which has  $k \geq 2$  disjoint channels. Let  $\{C_1, C_2, \dots, C_k\}$  and  $\{D_1, D_2, \dots, D_{k-1}\}$  be interlacing collections of channels of  $w$  and  $\text{fw}(w)$  (as described in Corollary 14.6). Let  $S = \mathbf{st}(w)$  and  $T = \mathbf{st}(\text{fw}(w))$ . Let  $r_0$  be the altitude of the stream from the class of  $T$  which is concurrent to  $S$ . Then*

$$a(T) = r_0 + h(C_1, C_2),$$

and

$$h(C_i, C_{i+1}) = h(D_{i-1}, D_i), \text{ for } 1 < i < k.$$

The theorem will be split into two cases; see Theorems 15.1 and 15.3 for proofs. Thus the vector  $\rho - \mathbf{r}$  from Section 5.2 is the vector of distances between channels.

## 9. SHI'S ALGORITHM AND KAZHDAN-LUSZTIG CELLS

There is a  $\mathbb{Z}$  action on the set of partial permutations called shifting the window: for a partial permutation (in window notation)  $[w_1, \dots, w_n]$  shifting the window by 1 gives the permutation  $[w_2, \dots, w_n, w_1 + n]$ .

Let  $w = [w_1, \dots, w_n]$  be a partial permutation in  $\tilde{A}_{n-1}$  in window notation. Recall the definition of a Knuth move.

**Definition 9.1.** Suppose  $w$  is a permutation. A permutation  $w'$  is obtained from  $w$  via a *Knuth move* if for some  $i \in \mathbb{Z}$ , Consider the following *combing* procedure originally introduced by Shi in [Shi91].

- $w'(i) = w(i+1)$ ,  $w'(i+1) = w(i)$ , and for all  $j \notin \bar{i} \cup \overline{i+1}$ ,  $w'(j) = w(j)$ , and
- $w(i-1)$  or  $w(i+2)$  is between  $w(i)$  and  $w(i+1)$ .

Consider the following *combing* procedure originally introduced by Shi in [Shi91].

- Let  $w_i$  be the first element such that  $w_{i-1} > w_i$ ;
- Let  $w_j$  be the first element with  $j < i$  such that  $w_i < w_j$ ;
- Replace  $w = [w_1, \dots, w_n]$  with

$$w' = [w_1, \dots, w_{j-1}, w_i, w_{j+1}, \dots, w_{i-1}, w_{i+1}, \dots, w_n, w_j + n],$$

obtained from  $w$  by a combination of Knuth moves and window shifts.

Now we can describe Shi's algorithm for affine insertion. While we want to start with  $w \in \tilde{A}_{n-1}$ , it is convenient to work in slightly greater generality: a partial permutation  $w$  such that in the window notation all the  $\emptyset$ 's are at the beginning. Let  $\ell_1, \ell_2, \dots$  be the Greene-Kleitman invariants of the Shi poset  $P_w$ , i.e.  $\ell_1$  is the size of the largest antichain,  $\ell_1 + \ell_2$  is the maximal size of union of two antichains, etc.

- Comb  $w$  until all elements become ordered increasingly (this does not change the position of the  $\emptyset$ 's).
- If the resulting window notation is  $[\emptyset, \dots, \emptyset, w_1, \dots, w_k]$  with  $w_{\ell_1} < w_1 + n$ , then remove the first  $\ell_1$  elements from  $w$  (put  $\emptyset$  in their place) and put their residue classes into the first row of tabloid  $P(w)$ ;
- Otherwise, replace  $w$  with  $[\emptyset, \dots, \emptyset, w_2, \dots, w_k, w_1 + n]$  and repeat starting with combing.
- Once a row has been removed, repeat the whole algorithm with the next  $\ell_i$  to remove the next row of  $P(w)$ , stop when all entries are  $\emptyset$ .

**Theorem 9.2.** [Shi91] *The left cells in type  $\tilde{A}_{n-1}$  are in bijection with tabloids filled with distinct residues modulo  $n$ . The algorithm described produces the tabloid corresponding to the left cell of  $w$ .*

*Example 9.3.* Let  $n = 7$  and take  $w = [7, 8, 18, 5, 2, 3, 13]$ . One can check that the partition of Greene-Kleitman invariants of the associated Shi poset  $P_w$  is  $(3, 2, 1, 1)$ . Applying the combing procedure we get

$$[7, 8, 18, 5, 2, 3, 13] \mapsto [5, 8, 18, 2, 3, 13, 14] \mapsto [2, 8, 18, 3, 13, 14, 12] \mapsto [2, 3, 18, 13, 14, 12, 15] \mapsto \\ [2, 3, 13, 14, 12, 15, 25] \mapsto [2, 3, 12, 14, 15, 25, 20] \mapsto [2, 3, 12, 14, 15, 20, 32].$$

Since  $2 + 7 < 12$ , we do not have the correct length  $\ell_1 = 3$  of the initial segment to peel. Therefore, we need to move 2 to the back and continue combing:

$$[2, 3, 12, 14, 15, 20, 32] \mapsto [3, 12, 14, 15, 20, 32, 9] \mapsto [3, 9, 14, 15, 20, 32, 19] \mapsto \\ [3, 9, 14, 15, 19, 32, 27] \mapsto [3, 9, 14, 15, 19, 27, 39].$$

Again,  $3 + 7 < 14$ , so we continue:

$$[3, 9, 14, 15, 19, 27, 39] \mapsto [9, 14, 15, 19, 27, 39, 10] \mapsto [9, 10, 15, 19, 27, 39, 21] \mapsto \\ [9, 10, 15, 19, 21, 39, 34] \mapsto [9, 10, 15, 19, 21, 34, 46].$$

At this point we can peel off the first  $\ell_1 = 3$  elements, as  $9 + 7 > 15$ , getting the first row of the tabloid to be  $\{\bar{9}, \bar{10}, \bar{15}\} = \{\bar{1}, \bar{2}, \bar{3}\}$ . In principle we may need to keep applying the algorithm to the result  $[\emptyset, \emptyset, \emptyset, 19, 21, 34, 46]$  at this point. However in this case we are



lucky and  $19 + 7 > 21$ , so the second row  $\{\overline{0}, \overline{5}\}$  peels off right away. Finally, we are left with  $[\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, 34, 46]$  which give us the third row of the tabloid  $\{\overline{6}\}$ , and the fourth row of the tabloid  $\{\overline{4}\}$ . Thus, at the end we get the result of Shi algorithm to be

$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{5}$	
$\overline{6}$		
$\overline{4}$		

**Theorem 9.4.** *The outcome of Shi's algorithm coincides with the tabloid  $P(w)$  of AMBC.*

The proof is found on page 40.

## Part 2. Proofs.

### 10. THE AFFINE COXETER GROUP $\overline{W} \subset \widetilde{W}$

The purpose of this section is to describe, in terms of conditions on a weight, when an extended affine permutation belongs to the actual affine Coxeter group  $\overline{W}$ .

**Theorem 10.1.** *Suppose  $w \in \widetilde{W}$  and  $\Phi(w) = (P, Q, \rho)$ . Then  $w \in \overline{W}$  if and only if  $\sum_i \rho_i = 0$ .*

The rest of the section is devoted to the proof.

**Definition 10.2.** Consider a partial permutation  $w$ . Let  $B_w^{restr}$  be the collection of balls of  $w$  in rows  $1, \dots, n$ . The (scaled) *center of gravity* is the integer

$$G_w = \frac{1}{n} \sum_{(i, w(i)) \in B_w^{restr}} w(i) - i.$$

Thus  $G_w$  is an “average” of the diagonals of all  $(n, n)$ -translate classes.

Note that by definition, for  $w \in \widetilde{W}$  we have  $w \in \overline{W}$  if and only if  $G_w = 0$ . The same definition works for any collection of cells which is preserved by translation by  $(n, n)$ , for example a stream.

**Lemma 10.3.** *Suppose  $w$  is a partial permutation. Then  $G_w = G_{fw(w)} + G_{st(w)}$ .*

*Proof.* Let us analyze one of the Young diagrams in the step of AMBC (see Figure 15). The set of columns of the black balls is the same as the set of columns of the green balls and the  $*$ . Similarly for the sets of rows. Hence the sum of diagonals of black balls is equal to the sum of diagonals of green balls plus the diagonal of the  $*$ . Summing over the first  $m$  Young diagrams finishing the proof.  $\square$

Repeated application of the above lemma gives that if  $\Phi(w) = (P, Q, \rho)$ , where the tabloids and weight have  $\ell$  rows, then  $G_w = \sum_{i=0}^{\ell} G_{st_{\rho_i}(P_i, Q_i)}$ . Thus to prove theorem 10.1 it suffices to prove the following lemma.

**Lemma 10.4.** *If  $w \in \widetilde{W}$  and  $\Phi(w) = (P, Q, \rho)$ , where the tabloids and weight have  $\ell$  rows, then*

$$\sum_{i=0}^{\ell} G_{st_{\rho_i}(P_i, Q_i)} = \sum_{i=0}^{\ell} \rho_i.$$



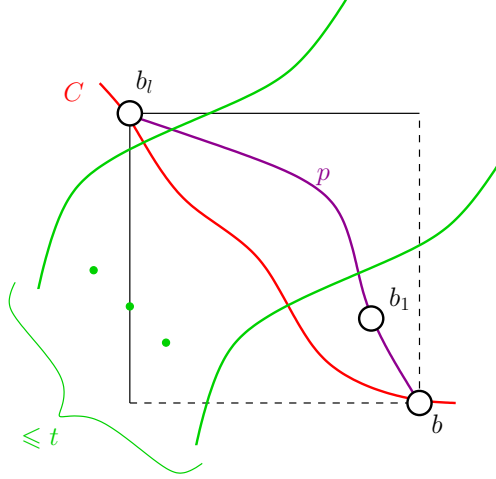


FIGURE 16. Proof of Proposition 3.10. The preimage of  $c$  is shown in green. The square referred to at the end of the proof is shown in black; it is “half-open” in the sense that the cells on the bottom and right boundary are not included while the rest of the boundary is included.

Number the balls of  $C$  consecutively; denote this numbering by  $\tilde{d} : C \rightarrow \mathbb{Z}$ . Recall that the channel numbering was defined for any  $b \in \mathcal{B}_w$  by

$$d_w^C(b) := \sup_{(b_0, b_1, \dots, b_k)} \tilde{d}(b_k) + k,$$

where the sum is over all paths from  $b$  to  $C$ .

First we will do a special case; namely we will show that the channel numbering coincides with  $\tilde{d}$  on the channel. The original statement will easily follow from this special case.

*Proof of Proposition 3.10.* It is tautologically true that  $d_w^C(b) \geq \tilde{d}(b)$ . Suppose, toward a contradiction, that  $d_w^C(b) > \tilde{d}(b)$ . Then there is a path from  $b$  to  $C$  of higher worth than a path to the same place along the channel. Extend this path along the channel to the nearest  $(n, n)$ -translate of  $b$ . Denote the resulting path by  $p = (b_0, \dots, b_l)$ ; its worth is  $\tilde{d}(b_l) + l$ . Then  $b_0 = b$  and  $b_l = b - t \cdot (n, n)$  for some  $t > 0$ . Moreover, the worth of the path along the channel is  $\tilde{d}(b_l) + tm$ , so  $l > tm$ .

By Dilworth’s Theorem, the Shi poset is a union of  $m$  chains. Consider the projections  $\pi(b_1), \dots, \pi(b_l)$  of the balls in the path. By pigeonhole principle, at least  $t + 1$  of the balls project to the same chain  $c$  of the Shi poset. The preimage of  $c$  under the projection consists of all the  $(n, n)$ -translates of a collection of balls which are pairwise in positive slope with each other. Each such translate contains at most one ball of the path, since any pair of balls in the path is negatively sloped. However only  $t$  such translates can possibly intersect the “half-open” square with corners at  $b$  and  $b_l$  (see Figure 16). This contradicts the fact that there are  $t + 1$  balls of the path projecting into  $c$ .  $\square$

Now it is easy to finish the general case.

*Proof of Proposition 3.9.* Notice that if  $b$  is northwest of  $b'$  then for any path from  $b$  to  $C$  there exists a path from  $b'$  to  $C$  of higher worth. Thus if  $d(b')$  is finite then  $d(b) < d(b')$  is

finite. Now every ball is northwest of some ball of  $C$ . Hence the proposition follows from Proposition 3.10.  $\square$

*Remark 11.2.* We note that while apriori computing a channel numbering is an infinite problem, one can, in fact, reduce it to a finite one. A path containing two  $(n, n)$ -translates of the same ball can be shortened without decreasing its worth. Hence we only need to consider finitely many paths.

**Corollary 11.3.** *Suppose  $(b_0, \dots, b_k)$  is a reverse path between two  $(n, n)$ -translates. Say  $b_k = b_0 + t \cdot (n, n)$  for some integer  $t \geq 0$ . Then  $k \leq tm$ .*

*Proof.* Suppose that is not the case. Pick a path  $P$  from  $b_k$  to  $C$ . Translate  $P$  to begin at  $b_0$  and form a new path from  $b_k$  to  $C$  by preceding the translated path with  $(b_k, \dots, b_0)$ . The resulting path has higher worth than  $P$ . Since  $P$  was arbitrary, this contradicts the conclusion of the proposition that the worth of paths is bounded above.  $\square$

The second paragraph of the proof of Proposition 3.10 has a nice consequence which we want to emphasize for later use.

**Lemma 11.4.** *Suppose for some  $t > 0$  we have a path  $(b_0, \dots, b_{tm})$  such that  $b_{tm} = b_0 - t \cdot (n, n)$ . Then for each  $i$  there exists a channel  $C$  such that  $b_i \in C$ .*

*Proof.* First suppose that the projections  $b_1, \dots, b_{tm}$  are distinct elements of  $P_w$ . Thus after taking projection we have  $tm$  elements of the Shi poset. No more than  $t$  of these may fall into a single chain (as we have seen in the second paragraph of the proof of Proposition 3.10), and each antichain has at most  $m$  elements. Hence these elements must split into  $t$  disjoint antichains of size  $m$ . By definition, all the balls in this case lie on the corresponding channels.

Now we need to get rid of the restriction that the projections of the elements of the path are distinct. The proof is by induction on  $t$ . The case  $t = 1$  works by definition. Suppose  $t > 1$ . By the previous paragraph, we can restrict to the case when there exists  $1 \leq i < j \leq tm$  such that  $\varphi(b_i) = \varphi(b_j)$ . Fix an innermost such pair of indices  $i, j$ , i.e. such that  $b_{i+1}, \dots, b_j$  project to distinct elements. Of course then  $b_j = b_i - t'(n, n)$  for some  $t'$ . One can see that  $j = i + t'm$ ; otherwise the path  $p = (b_0, \dots, b_i, b_{j+1} + t'(n, n), \dots, b_{tm} - t'(n, n))$  would have enough elements to contradict Corollary 11.3. By the first paragraph,  $b_i, \dots, b_j$  all lie on channels. By induction, all elements of  $p$  lie on channels. This finishes the proof since  $xb_k$  is an  $(n, n)$ -translate of one of the above balls.  $\square$

**11.3. Distance between channels and rivers.** In this section we are concerned with the notion of distance between channels from Definition 3.16. This notion will play a minor role in describing the structure of general proper numberings, and will be more profoundly relevant when studying the weight  $\rho(w)$ . The main result of the section is a proof that distance is indeed a pseudometric on the set of channels.

**Proposition 11.5.** *Suppose for some  $C_1, C_2 \in \mathcal{C}_w$ , neither  $C_1$  is southwest of  $C_2$  nor  $C_2$  is southwest of  $C_1$ . Then  $h(C_1, C_2) = 0$ .*

*Proof.* Let  $A_1$  and  $A_2$  be the maximal antichains corresponding to  $C_1$  and  $C_2$ , respectively. By Lemma 11.1, the set of minimal elements of  $A_1 \cup A_2$  forms a maximal antichain. Hence the set  $C$  of southwest-most elements of  $C_1 \cup C_2$  forms a channel. Since neither of the channels

is southwest of the other,  $C$  intersects both  $C_1$  and  $C_2$ . Since each channel is numbered by consecutive integers in any proper numbering, it follows easily that  $h(C_1, C_2) = 0$ .  $\square$

*Remark 11.6.* Consider  $C_1, C_2 \in \mathcal{C}_w$  such that  $h(C_1, C_2) \neq 0$ . Let  $a_1 = d_w^{C_1}$  and  $a_2 = d_w^{C_2}$  with shifts chosen so the two numberings coincide on  $C_1$ . Then we can show that for any  $b \in C_2$  we have  $a_2(b) > a_1(b)$ . In fact, for any proper numbering  $a_3$  which coincides with  $a_1$  on  $C_1$  and for any ball  $b$ , we must have  $a_3(b) \geq a_1(b)$ . This follows from considering the path of maximal worth from  $b$  to  $C_1$ ; the values of  $a_3$  on each step of the path must change by at least 1. In case of  $a_2$ , equality is forbidden by the condition that the distance between  $C_1$  and  $C_2$  is nonzero.

Now we can show that distance is a pseudometric on channels.

*Proof of Proposition 3.17.* It is clear that  $h$  is nonnegative and symmetric, so we only need to check the triangle inequality. Suppose  $C_1, C_2$ , and  $C_3$  are channels. We want to show that  $h(C_1, C_3) \leq h(C_1, C_2) + h(C_2, C_3)$ . If any of the pairwise distances are 0, then the statement is clear. Thus assume that is not the case.

Since  $C_1$  and  $C_3$  play interchangeable roles and the situation remains the same if we reflect with respect to the main diagonal, up to symmetry there are only two cases to consider. First, if  $C_1$  is the southwest of the three channels and  $C_2$  is the middle one. In this case it is easy to see that equality holds. Second, if  $C_1$  is the southwest of the three channels and  $C_2$  is the northeast one. In this case  $h(C_1, C_2) = h(C_1, C_3) + h(C_3, C_2)$ , and a strict inequality holds.  $\square$

A pseudometric on a set naturally partitions it into equivalence classes, which in this case were called rivers.

To conclude this section we give a criterion of when a monotone numbering coincides with a channel numbering.

*Remark 11.7.* Consider a partial permutation  $w$ , a channel  $C$ , its corresponding channel numbering  $d_w^C$ . Suppose we have another increasing numbering  $d$  which coincides with  $d_w^C$  on  $C$ . We will now show that for a ball  $b$  we have  $d(b) = d_w^C(b)$  precisely when there is a path  $(b = b_0, b_1, \dots, b_l)$  to  $C$  such that  $d(b_{i+1}) = d(b_i) - 1$ . Indeed, since there necessarily exists a path  $(b = b'_0, b'_1, \dots, b'_l)$  to  $C$  such that  $d_w^C(b_{i+1}) = d_w^C(b_i) - 1$ , both inequalities  $d_w^C(b) \leq d(b)$  and  $d_w^C(b) \geq d(b)$  follow from the fact that the two numberings are monotone.

**11.4. General proper numberings.** In this section we describe the structure of general proper numberings. First we point out that not every proper numbering is derived from a channel.

*Example 11.8.* For the permutation  $[6, 1, 8, 3, 10, 5]$  there are only two channels, however there are three proper numberings (see Figure 17).

Now we show that any proper numbering is semi-periodic with period equal to the width of the Shi poset.

*Proof of Proposition 3.4.* Let  $m$  be the width of  $P_w$ . We want to argue that for any  $b \in \mathcal{B}_w$  we have  $d(b + (n, n)) = d(b) + m$ , given that  $d$  satisfies Monotonicity and Continuity. The proof will consist of a sequence of claims.

**Claim.** For any  $b \in \mathcal{B}_w$  we have

$$d(b + (n, n)) \leq d(b) + m.$$

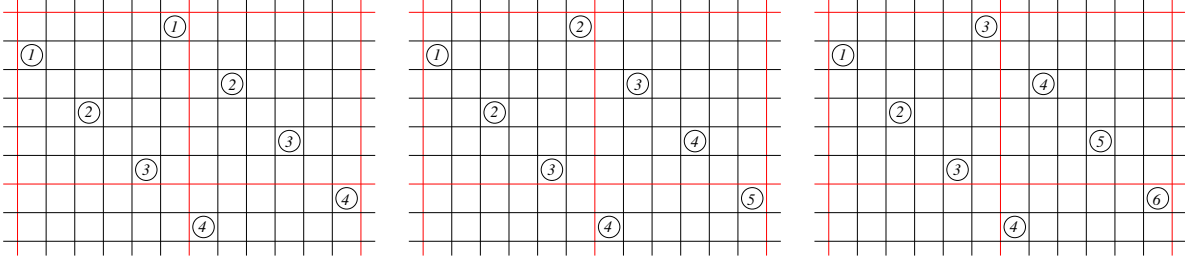


FIGURE 17. Three proper numberings of  $[6, 1, 8, 3, 10, 5]$ .

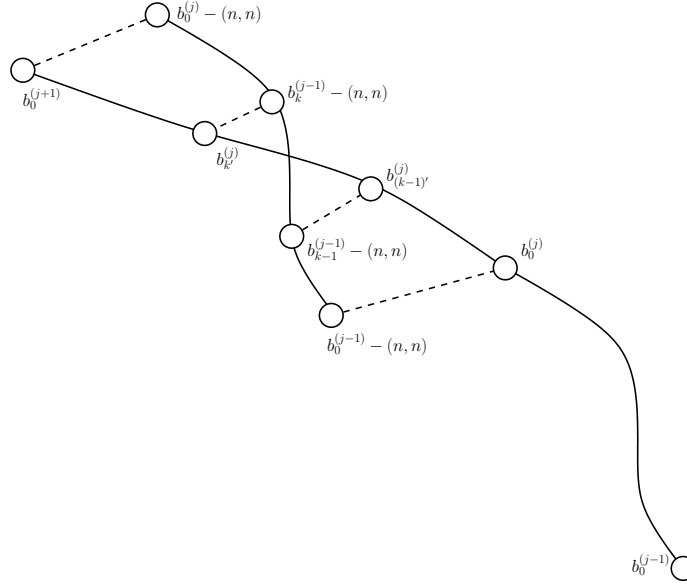


FIGURE 18. First claim in proof of Proposition 3.4.

Toward a contradiction, assume that for some  $b$  we have  $d(b - (n, n)) < d(b) - m$ . By Continuity, one can find a path

$$(b = b_0^{(0)}, b_1^{(0)}, \dots, b_M^{(0)} = b_0^{(1)})$$

such that  $d(b_{i+1}^{(0)}) = d(b_i^{(0)}) - 1$  and  $d(b_0^{(1)}) = d(b - (n, n))$ . According to our assumption we have  $M > m$ . Informally, we made sufficiently many size 1 steps in northwest direction from  $b$  to end up with the same number of the final ball  $b_0^{(1)}$  as that of  $b - (n, n)$ . The number of steps  $M$  that we made must be larger than  $m$  by our assumption.

Let us look now at the path

$$(b_0^{(0)} - (n, n), b_1^{(0)} - (n, n), \dots, b_0^{(1)} - (n, n)).$$

By Monotonicity, for each  $i$  we have  $d(b_{i+1}^{(0)} - (n, n)) \leq d(b_i^{(0)} - (n, n)) - 1$ . Therefore,

$$d(b_0^{(1)}) - M = d(b_0^{(0)} - (n, n)) - M \geq d(b_0^{(1)} - (n, n)).$$

Thus, the gap in numbering between  $b_0^{(1)}$  and  $b_0^{(1)} - (n, n)$  is at least as large as that between  $b_0^{(0)}$  and  $b_0^{(0)} - (n, n)$ , and in particular this gap is greater than  $m$ .

Now we can repeat the same construction building a new pair  $b_0^{(2)}$  and  $b_0^{(2)} - (n, n)$ , etc. On each step we get the gap between  $b_0^{(j)}$  and  $b_0^{(j)} - (n, n)$  to be strictly larger than  $m$ .

Let us compare locations of  $b_0^{(j)}$  and  $b_0^{(j-1)} - (n, n)$ . First, they must be distinct. Indeed, otherwise the path we built between  $b_0^{(j-1)}$  and  $b_0^{(j-1)} - (n, n)$  has length strictly greater than  $m$ , which is impossible since any such path projects to an antichain in the Shi poset. Furthermore,  $b_0^{(j)}$  and  $b_0^{(j-1)} - (n, n)$  must be not comparable in the  $NW$  order, since they have the same numbering and monotonicity holds. Therefore,  $b_0^{(j)}$  is either northeast or southwest of  $b_0^{(j-1)} - (n, n)$ . We claim that it is impossible for each  $j = 1, 2, \dots$  the answer to be the same: either always northeast, or always southwest. Indeed, if that was the case, we would be getting balls  $b_0^{(j)} = (a_j, b_j)$  with  $a_j - b_j$  either indefinitely increasing or indefinitely decreasing. Those differences take only a finite set of values however.

Therefore, there exists a  $j$  such that  $b_0^{(j)}$  is northeast of  $b_0^{(j-1)} - (n, n)$ , while  $b_0^{(j+1)}$  is southwest of  $b_0^{(j)} - (n, n)$ , or vice versa. Since the two cases are similar, let us assume it is the former. Compare the paths

$$p = (b_0^{(j)}, b_1^{(j)}, \dots, b_0^{(j+1)}) \text{ and}$$

$$q = (b_0^{(j-1)} - (n, n), b_1^{(j-1)} - (n, n), \dots, b_0^{(j)} - (n, n))$$

that connect them. For each  $i$  within the range let  $i'$  be the index such that  $d(b_i^{(j-1)} - (n, n)) = d(b_{i'}^{(j)})$ , such  $i'$  definitely exists because the first chain was constructed using Continuity. Let  $k$  be the first index such that  $b_k^{(j-1)} - (n, n)$  lies strictly northeast of  $b_{k'}^{(j)}$ . Then it is easy to see that  $b_{(k-1)'}^{(j)}$  is southeast of  $b_k^{(j-1)} - (n, n)$  and we have a path

$$(b_0^{(j)}, \dots, b_{(k-1)'}^{(j)}, b_k^{(j-1)} - (n, n), \dots, b_0^{(j)} - (n, n))$$

of length greater than  $m$  (since the length is at least that of  $q$ ). This is a contradiction since such path projects to an antichain in the Shi poset.

**Claim.** Let  $b \in C$  be a ball in a channel  $C \in \mathcal{C}_w$ . Then we have

$$d(b + (n, n)) = d(b) + m.$$

Indeed,  $d(b + (n, n)) \geq d(b) + m$  since the path from  $b + (n, n)$  to  $b$  along the channel has  $m$  steps. Combined with the previous claim, the needed result follows.

**Claim.** For any  $b \in \mathcal{B}_w$  call the pair  $b$  and  $b + (n, n)$  a *gap* if we have  $d(b + (n, n)) < d(b) + m$ . Then for any  $b$  there is only a finite number of gaps of the form

$$b + k(n, n) \text{ and } b + (k + 1)(n, n)$$

as  $k$  ranges over  $\mathbb{Z}$ . To see this, consider a channel  $C \in \mathcal{C}_w$ , and consider ball  $\bar{b} \in C$  such that  $d(\bar{b}) = d(b)$ . If  $\bar{b} = b$ , there is nothing to prove by the previous claim, as there are no gaps at all. There exists  $\ell$  such that  $b$  is southeast of  $\bar{b} - \ell(n, n)$ . Then if for a large enough  $k$  we have more than  $\ell m$  gaps between  $b$  and  $b + k(n, n)$ , we arrive at a contradiction with monotonicity:

$$d(b + k(n, n)) < d(b) + km - \ell m = d(\bar{b}) + km - \ell m = d(\bar{b} + (k - \ell)(n, n)),$$

while  $b + k(n, n)$  is southeast of  $\bar{b} + (k - \ell)(n, n)$ . Similarly one arrives at a contradiction assuming there exists a large number of gaps between  $b$  and  $b - k(n, n)$ . This means that the total number of gaps has to be finite.

**Claim.** There exists a  $K$  such that for  $d(b) < K$  there is no gap between  $b$  and  $b - (n, n)$ . Indeed, there is a finite number of balls non-equivalent under projection  $\varphi_w$ . For each of them there is  $k$  such that there are no gaps between  $b - i(n, n)$  and  $b - (i + 1)(n, n)$  for  $i > k$ . Then one just needs to choose  $K$  small enough to be smaller than  $d(b - k(n, n))$  for all of finitely many equivalence classes of  $b$ .

**Claim.** For any  $b \in \mathcal{B}_w$  we have

$$d(b + (n, n)) = d(b) + m.$$

Indeed, by continuity we can build an infinite path

$$b = (b_0, b_1, b_2, \dots)$$

with  $d(b_{i+1}) = d(b_i) - 1$ . Eventually we will have  $d(b_i) < K$ , where  $K$  is as in the previous claim. Furthermore, eventually we will find  $b_i$  and  $b_j$  such that  $\varphi_w(b_i) = \varphi_w(b_j)$  and  $d(b_j) < d(b_i) < K$ . Since there are no gaps below the label  $K$ , we conclude that  $b_i = b_j + k(n, n)$ , where  $k = \frac{j-i}{m}$ . Consider in that case the following path:

$$(b + k(n, n), b_1 + k(n, n), \dots, b_j + k(n, n) = b_i.)$$

By Monotonicity, we must have

$$d(b + k(n, n)) \geq j + d(b_i) = j + d(b_j) + (j - i) = d(b) + (j - i).$$

This means that there are no gaps between  $b$  and  $b + k(n, n)$ , and thus in particular there is no gap between  $b$  and  $b + (n, n)$ . The claim follows, which means that Proposition 3.4 holds.  $\square$

In the process (second claim) we have shown that a proper numbering numbers any channel by consecutive integers, namely we proved Lemma 3.15. Continuing our study of the structure theory of proper numberings we show that a proper numbering is determined by its value on the channels.

*Proof of Proposition 3.12.* Choose a ball  $b$ , and form a path  $p = (b_0 = b, b_1, \dots)$  such that and  $d(b_{i+1}) = d(b_i) - 1$ . By pigeonhole principle, the projections of two balls of  $p$  must coincide. Thus by Lemma 11.4,  $p$  necessarily leads to some channel. Thus  $d(b)$  is the worth of a path from  $b$  to some channel. However  $d(b)$  is greater than or equal to the worth of any path from  $b$  to any channel since the value of  $d$  must increase at each step of such path. Hence  $d(b)$  must be equal to the maximal worth of a path from  $b$  to one of the channels. The same statement can be made about  $d'$ . Hence  $d = d'$ .  $\square$

In the rest of the section we discuss when does a numbering of all channels of  $w$  extend to a proper numbering. While we do not need this for future results in this paper, it finishes our classification of proper numberings.

**Definition 11.9.** Suppose  $w$  is a partial permutation and  $\tilde{d} : \bigcup_{C \in \mathcal{C}_w} C \rightarrow \mathbb{Z}$  is a numbering which restricts to a proper numbering on each channel. Fix a shift of  $d_w^{C_1}$ . We say that  $\tilde{d}$  is *consistent on channels* if for any  $C_1, C_2 \in \mathcal{C}_w$  and some (equiv. any)  $b \in C_2$  we have

$$d_w^{C_1}(b) \leq \tilde{d}(b) \leq d_w^{C_1}(b) + h(C_1, C_2).$$



**Proposition 11.10.** *Suppose  $w$  is a partial permutation and  $\tilde{d} : \bigcup_{C \in \mathcal{C}_w} C \rightarrow \mathbb{Z}$ . Then there exists a proper numbering which restricts to  $\tilde{d}$  if and only if  $\tilde{d}$  is consistent on channels.*

*Proof.* First, we will show that if  $\tilde{d}$  is not consistent on channels, then  $\tilde{d}$  is not a restriction of a proper numbering. Suppose  $d$  is a proper numbering. The value of  $d$  must decrease at every step of a path of maximal worth from  $b$  to  $C_1$ . Hence  $d_w^{C_1}(b) \leq d(b)$ . Now pick  $d_w^{C_2}$  to coincide with  $d_w^{C_1}$  on  $C_1$ . By definition  $d_w^{C_2}(b) = d_w^{C_1}(b) + h(C_1, C_2)$ . Consider a maximal worth path from  $C_1$  to  $b$ . The value of  $d$  must decrease at every step of this path so  $d(b) \leq d_w^{C_2}(b)$ . This demonstrates that the condition of  $\tilde{d}$  being consistent on channels is necessary.

Now suppose  $\tilde{d}$  is consistent on channels. Then for any  $b$  let

$$d(b) := \sup_{(b_0=b, b_1, \dots, b_k)} \tilde{d}(b_k) + k,$$

where the supremum is taken over all paths from  $b$  to channels of  $w$ . The resulting numbering is always well defined regardless of the assumption on  $\tilde{d}$ . Since  $\tilde{d}$  restricts to a proper numbering on each channel,  $d$  is proper. The inequalities on  $\tilde{d}$  ensure that for any  $b$  in any channel we have  $d(b) = \tilde{d}(b)$ , i.e. that  $d$  restricts to  $\tilde{d}$  on the channels.  $\square$

*Remark 11.11.* The two inequalities in the above proposition are superfluous in that one of them is obtained from the other by reversing the roles of  $C_1$  and  $C_2$ . The reason we presented them in this way was to make an algorithmic construction of all proper numberings possible as follows. Fix a proper numbering on the southwest channel  $C_1$ . Find the southwest channel  $C_2$  which does not intersect  $C_1$ . Assign to it any of the numberings allowed by the proposition. Find the southwest channel  $C_3$  which does not intersect  $C_1$  or  $C_2$ . Assign to it any numbering that is consistent with both  $C_1$  and  $C_2$ . Repeat until no new channel can be found; this will define a numbering consistent on channels.

**11.5. Stabilization.** In this section we show that the numbering scheme from the non-affine Matrix-Ball Construction with an arbitrary starting condition will eventually stabilize to a proper numbering.

First, we describe what we mean by the above numbering scheme. For each  $z \in P_w$  pick a ball  $b_z$  in  $\varphi^{-1}(z)$ ; only the balls  $\{b_z + k(n, n) : z \in P_w, k \geq 0\}$  will be numbered. For example balls below a horizontal line between the 0-th and 1-st rows would do. Number these balls as in non-affine MBC. An example is shown in Figure 19. Denote the resulting numbering by  $d_{cut}$ .

It is clear that the resulting numbering satisfies the Monotonicity and Continuity properties when defined. In fact, even more is true:

**Theorem 11.12.** *There exists a constant  $c \in \mathbb{Z}^{>0}$  such that for any ball  $b \in \mathcal{B}_w$  with  $d_{cut}(b) \geq c$ , we have  $d_{cut}(b + (n, n)) = d_{cut}(b) + m$ .*

*Proof.* Notice that in the numbering  $d_{cut}$ , any pair of balls with the same number are related in the NE partial order. Modulo translation by multiples of  $(n, n)$ , there are finitely many such sequences. Hence there are numbers  $c, c', t \in \mathbb{Z}^{>0}$  such that  $d_{cut}^{-1}(c')$  is formed by translating  $d_{cut}^{-1}(c)$  by  $t(n, n)$ . Let  $m' = c' - c$ . By construction, if  $d_{cut}(b) \geq c$ , then  $d_{cut}(b + t(n, n)) = d_{cut}(b) + m'$ . Consider a new numbering  $\tilde{d}$  which coincides with  $d_{cut}$  on balls labeled  $c$  or above, and is extended to the remaining balls according to the rule  $\tilde{d}(b - t(n, n)) = \tilde{d}(b) - m'$ . It is clear that  $\tilde{d}$  is a proper numbering.  $\square$

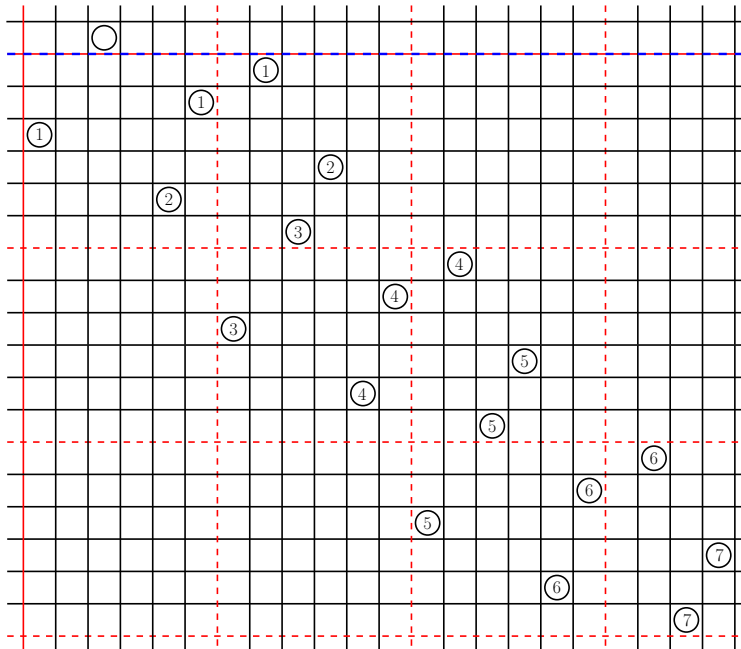


FIGURE 19. An example of the cutoff numbering  $d_{cut}$ . The cutoff line is the dashed blue horizontal line between the 0-th and 1-st rows.

## 12. TABLOIDS

In this section we prove two interesting results about the tabloids obtained in AMBC. When describing the algorithm we used the southwest channel numbering at every step. The first result is that, in fact, if we use any proper numbering at every step, then the tabloids  $P(w)$  and  $Q(w)$  will be the same. The second result is that the tabloid  $P(w)$  is actually the same as the tabloid obtained from Shi's algorithm.

**12.1. Independence of the proper numbering choices.** The argument for the first main result is asymptotic in nature. We will be inserting via regular Robinson-Schensted algorithm larger and larger pieces of an affine permutation. We start by introducing necessary notation and proving a couple technical results about the behavior of the Robinson-Schensted algorithm in this scenario.

We deal with partial non-affine permutations (PNAP), namely sequences  $(a_1, \dots, a_k)$  where each  $a_i \in \mathbb{Z} \cup \{\emptyset\}$  and all the integers present in the sequence are distinct. We can apply the Robinson-Schensted bumping algorithm to such a sequence: every time we run into a  $\emptyset$  we don't do anything to the tableaux, but increment the next entry to be added to the recording tableau. We can also view a PNAP as a matrix in our usual way and apply the (non-affine) MBC. The tableaux resulting from the two algorithms must coincide; this reduces to the statement that MBC provides an implementation of the Robinson-Schensted correspondence combined with the standardization procedure for turning an PNAP into a permutation.

For a partial affine permutation  $w$ , let  $a_w$  be the PNAP  $(w(1), w(2), \dots, w(n))$ . Similarly, let  $a_w^k$  be the PNAP  $(w(1), w(2), \dots, w(kn))$ .

**Lemma 12.1.** *Let  $w$  be a partial affine permutation. The number of rows of  $P(a_w^k)$  is bounded above by a constant  $c_w$ , which is independent of  $k$ .*

*Proof.* Index the diagonals of the matrix of  $w$  so that the principal one is given the index 0 (so the ball  $(i, j)$  is on the  $(j - i)$ -th diagonal). Let  $d_{min}$  (resp.  $d_{max}$ ) be the minimal (resp. maximal) diagonal of a ball of  $a$ . Since all the balls for  $w$  are  $(n, n)$ -translates of those in the first  $n$  rows, for any  $k$ , all the balls of  $a^k$  lie (weakly) between these diagonals. Hence the balls of  $\text{fw}(w)$  must lie (weakly) between diagonals  $d_{min} + 1$  and  $d_{max} - 1$ . So after  $\left\lfloor \frac{d_{max} - d_{min}}{2} \right\rfloor + 1$  steps all the balls will disappear. Hence choosing

$$c_w = \left\lfloor \frac{d_{max} - d_{min}}{2} \right\rfloor + 1$$

will be sufficient. □

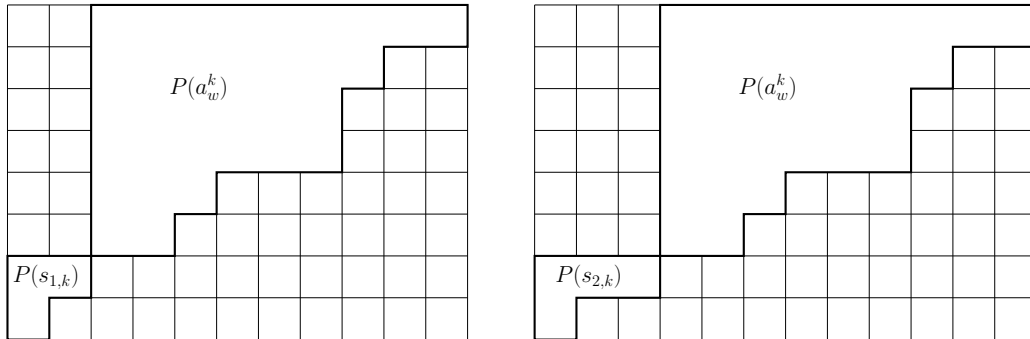
For (skew)-tableaux  $T_1, T_2$ , define  $h(T_1, T_2)$  as number of integers which lie in different rows in  $T_1$  and in  $T_2$  (if an integer is only in one of the tableaux then it is considered to lie in different rows; if an integer is not in either of the tableaux then it is considered to lie in the same rows). This is a pseudometric on the set of skew-tableaux and a metric on the set of tabloids. The next result shows that if a sequence of PNAPs approximates  $\{a_w^k\}$ , then the Robinson-Schensted insertion tableaux approximate  $\{P(a_w^k)\}$ .

**Lemma 12.2.** *Suppose  $w$  is a partial affine permutation. Let  $\{s_{1,k}\}, \{s_{2,k}\}, \{f_{1,k}\}, \{f_{2,k}\}$  be sequences of PNAPs such that  $p_{1,k} := s_{1,k}a_w^k f_{1,k}$  and  $p_{2,k} := s_{2,k}a_w^k f_{2,k}$  are PNAPs for any  $k > 0$ . Moreover, suppose that the sizes of the permutations in the four sequences are uniformly bounded by a number  $M$ , i.e.  $|s_{1,k}| \leq M, |s_{2,k}| \leq M, |f_{1,k}| \leq M, |f_{2,k}| \leq M$ . Then  $h(P(p_{1,k}), P(p_{2,k}))$  is bounded above by a constant independent of  $k$ ; more precisely*

$$h(P(p_{1,k}), P(p_{2,k})) \leq 2M + 2M \cdot c_w(M + c_w) + 2M(c_w + 2M + 1),$$

where  $c_w$  is the constant from Lemma 12.1.

*Proof.* Consider the insertion tableaux of  $s_{1,k}a_w^k$  and  $s_{2,k}a_w^k$ . By the jeu-de-taquin theory they are the unique straight-shaped tableaux of the jeu-de-taquin classes of:



The distance  $h$  between the two skew-tableaux is at most  $2M$ .

The number of jeu-de-taquin slides which need to be performed is equal to the area of the top-left rectangle in the figure. The width of that rectangle is  $\leq M$  and the height is  $\leq c_w$  (Lemma 12.1). So at most  $M \cdot c_w$  slides are necessary. The number of elements whose row is

changed by a slide is at most the height of the skew-tableau, which in our case is bounded by  $c_w + M$ . Thus

$$h(P(s_{1,k}a_w^k), P(s_{2,k}a_w^k)) \leq 2M + 2M \cdot c_w(M + c_w).$$

Now we analyze what happens as we insert the ending sections via the bumping algorithm. As each element is inserted, the number of rows increases by at most 1. Since the number of rows of  $P(s_{1,k}a_w^k)$  (and  $P(s_{2,k}a_w^k)$ ) is at most  $c_w + M$  and there are at most  $M$  insertions, the number of rows during the process remains at most  $c_w + 2M$ . Thus each insertion changes the row of at most  $c_w + 2M + 1$  integers. Hence

$$h(P(p_{1,k}), P(p_{2,k})) \leq 2M + 2M \cdot c_w(M + c_w) + 2M(c_w + 2M + 1).$$

□

We move on to the proof that the construction of a tabloid as in AMBC, but choosing various proper numberings in the process, is actually independent of the choices. The idea is that as we take  $a_w^k$  for larger and larger values of  $k$  and insert them using regular Robinson-Schensted correspondence, the residue classes of the entries by which the tableau grows with each additional step eventually stabilize. Moreover, given a sequence of choices of proper numberings, they can be shown to stabilize to the tabloid corresponding to this choice sequence.

Let  $d_1$  be a proper numbering of  $w$ ,  $d_2$  a proper numbering of  $w_2 := \text{fw}_{d_1}(w)$ ,  $d_3$  a proper numbering of  $w_3 := \text{fw}_{d_2}(w_2)$ , etc. If the last of the numberings is  $d_l$ , we call the sequence  $d = (d_1, \dots, d_l)$  a *p-sequence*. For a p-sequence  $d$  we can form a tabloid  $P(d)$  as in AMBC but with proper numbering choices of  $d$ . We will now show that  $P(d)$  actually depends only on  $w$ .

*Proof of Proposition 3.34 and Theorem 7.3.* We consider two related sequences of PNAPs, depending on a parameter  $k$  which should be thought of as large. The first is  $p_{1,k} = a_w^k$ . The second,  $p_{2,k}$ , consists only of those balls of  $p_{1,k}$  for which all the balls of  $w$  with the same value of  $d_1$  are also part of  $p_{1,k}$ . The balls to be discarded are circled in blue in Figure 20; the numbering shown is  $d_1$ .

Index the diagonals so that the diagonal of a cell  $(i, j)$  is indexed  $j - i$ . Let  $d_{\max}$  (resp.  $d_{\min}$ ) be the highest (resp. lowest) index of a diagonal of a ball of  $w$ . Let  $t_w := d_{\max} - d_{\min}$ . The number of rows containing balls with a given value of  $d_1$  is bounded above by  $t_w$ . Hence the total number of balls discarded from the top part of  $p_{1,k}$  to get  $p_{2,k}$  is at most  $t_w$ . Similarly for the bottom part. By Lemma 12.2,  $d(P(p_{1,k}), P(p_{2,k})) < c'_w$  for some constant  $c'_w$  which depends only on  $w$  (not on  $k$ ). Notice that  $p_{2,k}$  depends on  $d$ , so  $P(p_{2,k})$  depends on  $d$ . However  $p_{1,k}$  does not depend on  $d$ , so  $P(p_{1,k})$  also does not depend on  $d$ .

Let us first look in more detail at  $P(p_{1,k})$ , and the tabloid  $P'(p_{1,k})$  obtained from it by taking residues modulo  $n$  and forgetting about ordering inside rows. For  $k$  large enough, the MBC numbering stabilizes to a proper numbering as in Theorem 11.12. Say, when the MBC numbering is above  $M$  then it is proper. When  $k$  is larger yet, the additional balls of  $p_{1,k+1}$  will all be numbered greater than  $M$ . In this case one can see that the difference between the first rows  $P'(p_{1,k})$  and  $P'(p_{1,k+1})$  is exactly the same as the difference between the first rows  $P'(p_{1,k+1})$  and  $P'(p_{1,k+2})$ . Hence when  $k$  is large enough, then increasing  $k$  by 1 appends some fixed subset of  $[n]$  to the first row of the insertion tabloid  $P'$ . The same argument can be applied to  $\text{fw}(p_{1,k})$ , which becomes symmetric under  $(n, n)$ -translates after

the MBC numbering of  $p_{1,k+1}$  stabilized. Since the number of rows is bounded by a constant independent of  $k$ , we conclude that for  $k$  large enough, the difference between  $P'(p_{1,k})$  and  $P'(p_{1,k+1})$  is some fixed tabloid independent of  $k$ .

Now let us look at the first row of  $P(p_{2,k})$ . Here the MBC numbering coincides (up to an overall shift) with the numbering  $d_1$ . Hence the first row of  $P(p_{2,k})$  consists of column numbers of the southwest ball with a given number according to  $d_1$ . If  $m$  is the width of the Shi poset, and  $a_1, \dots, a_m$  are the first  $m$  of these column numbers then the first row of  $P(p_{2,k})$  looks like

$$a_1, \dots, a_m, a_1 + n, \dots, a_m + n, a_1 + 2n, \dots, a_m + 2n, \dots$$

So the first row of the tabloid  $P'(p_{2,k})$  obtained by passing to residues modulo  $n$  and disregarding the ordering within rows consists of  $\overline{a_1}, \dots, \overline{a_m}$  each one repeated a number of times (approximately  $k$ ). We claim that for  $k$  sufficiently large, the difference between the first rows of  $P'(p_{2,k})$  and  $P'(p_{2,k+1})$  is just  $\{\overline{a_1}, \dots, \overline{a_m}\} = P(d)_1$ .

We can also see that for sufficiently large  $k$ , the differences between the first rows  $P'(p_{2,k})$  and  $P'(p_{2,k+1})$  is the same as between the first rows of  $P'(p_{2,k-1})$  and  $P'(p_{2,k})$ . Indeed, this happens when the  $d_1$ -numbering of the new balls is larger than that of any of the balls dropped from the northwest corner. Of course, between  $P'(p_{2,k})$  and  $P'(p_{2,k+1})$  no residue class could be added twice. Moreover the number of times each residue class from  $\{\overline{a_1}, \dots, \overline{a_m}\}$  appears must increase with  $k$ . Thus the difference between the first rows of  $P'(p_{2,k})$  and  $P'(p_{2,k+1})$  is precisely  $P(d)_1$ .

Since, regardless of  $k$ , the first row of  $P(p_{1,k})$  differs by at most  $c'_w$  entries from the first row of  $P(p_{2,k})$ , we conclude that for  $k$  sufficiently large, the difference between the first rows of  $P'(p_{1,k})$  and  $P'(p_{1,k+1})$  is  $P(d)_1$ . This proves the statement of Theorem 12.1 for the first rows and at the same time concludes that  $P(d)_1$  only depends on  $w$  (since  $P'(p_{1,k+1})$  is manifestly independent of  $d$ ).

Consider sequences of PNAPs  $p_{1,k}^{(2)} = \text{fw}(p_{2,k})$  and  $p_{2,k}^{(2)}$  which is formed from  $p_{1,k}^{(2)}$  as before by dropping the incomplete layers of the  $d_2$  numbering. By the same argument as above we know that for sufficiently large  $k$ , the difference between the first rows of  $P'(p_{1,k}^{(2)})$  and  $P'(p_{1,k+1}^{(2)})$  is  $P(d)_2$ . Of course the first row of  $P'(p_{1,k}^{(2)})$  is by definition the second row of  $P'(p_{2,k})$ . Since, regardless of  $k$ , the second row of  $P(p_{1,k})$  differs by at most  $c'_w$  entries from the second row of  $P(p_{2,k})$ , we conclude that for  $k$  sufficiently large, the difference between the second rows of  $P'(p_{1,k})$  and  $P'(p_{1,k+1})$  is  $P(d)_2$ .

Repeating the argument for the remaining rows (whose number is bounded above by a constant independent of  $k$ ) finishes the proof.  $\square$

From now on we can write  $P(w)$  instead of  $P(d)$ .

**12.2. Equivalence to Shi's algorithm.** The first half of Shi's algorithm consisted of a series of Knuth moves, window shifts, and, when working with partial permutations, moving the  $\emptyset$ 's to the front; these operations were used to bring the permutation to a combed form. It is obvious that window shifts preserve the tabloid  $P(w)$  since they amount to shifting all balls north or south by 1 (of course they do affect  $Q(w)$ , but that is of no relevance in this section). Similarly collecting all  $\emptyset$ 's at the beginning of the window does not affect  $P(w)$ . We will show that Knuth moves also preserve  $P(w)$ . After that we will describe the southwest channel numbering of a combed permutation. We will see that taking the initial

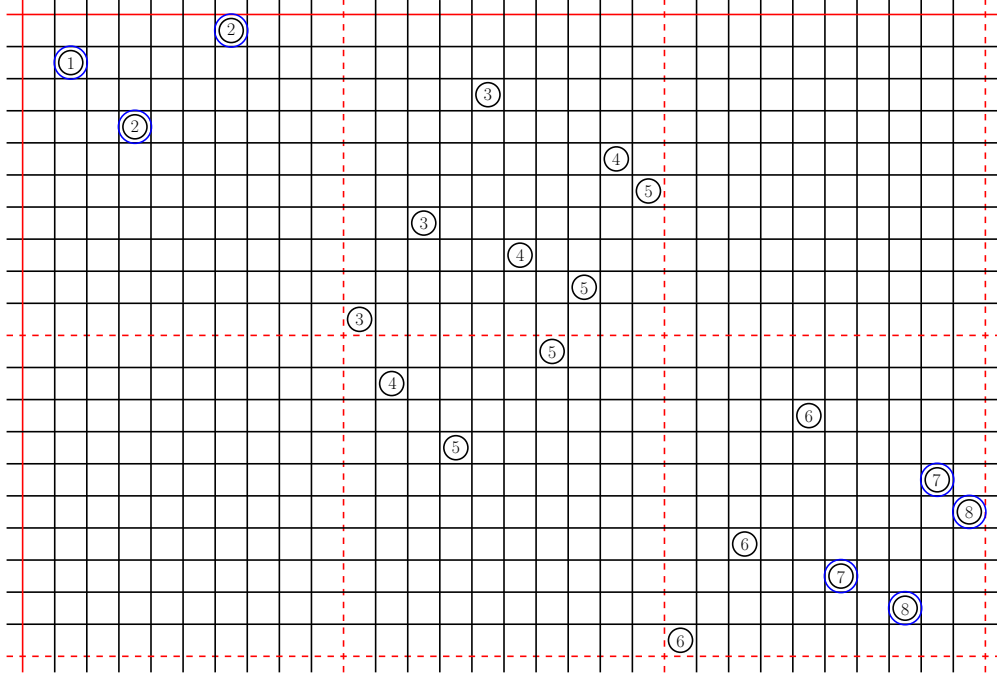


FIGURE 20. Balls of  $p_{1,k}$ ; to get  $p_{2,k}$  the blue balls need to be discarded.

maximal antichain and placing it in the row of the tabloid corresponds to the operation of forming a row of  $P(w)$  in AMBC. Moreover, we will see that the step of replacing the initial maximal antichain with  $\emptyset$ 's corresponds exactly to a step of AMBC.

**Lemma 12.3.** *Suppose  $w$  differs from  $w'$  by a Knuth move. Then  $P(w) = P(w')$ .*

*Proof.* Consider the PNAPs  $a_w^k$  and  $a_{w'}^k$ . Then  $a_w^k$  and  $a_{w'}^k$  differ by a sequence of ordinary Knuth moves as well as possibly by the first and last two entries. It is well known that non-affine Knuth moves preserve the insertion tableau, so Lemma 12.2 guarantees that the distance between  $P(a_w^k)$  and  $P(a_{w'}^k)$  is bounded above independently of  $k$ . The asymptotic interpretation of  $P(w)$  given in Theorem 7.3 finishes the proof.  $\square$

Now we describe the southwest channel numbering of a combed permutation (recall the definition from Section 9).

**Lemma 12.4.** *Suppose  $w$  is a combed (affine) partial permutation. Define a numbering on the balls in rows 1 through  $n$  by  $d(i, w(i)) = i$  (see Figure 21). Extend this to all balls by periodicity, i.e.  $d((i, w(i)) + k(n, n)) = i + km$ . Then  $d$  is the southwest channel numbering.*

*Proof.* First, let us show that  $d$  is a proper numbering. It is easy to see that Continuity holds; the only potential problem is Monotonicity. It is sufficient to show that for each  $i$ , we have  $d^{-1}(i)$  is a chain in the  $\leq_{sw}$  partial ordering. Indeed, if  $b$  is northwest of  $b'$  and  $d(b) \geq d(b')$  then by Continuity we can find  $b''$  northwest of  $b$  with  $d(b'') = d(b')$ ; of course  $b''$  and  $b'$  are incomparable with respect to  $\leq_{sw}$ . By periodicity, it is sufficient to consider  $1 \leq i \leq n$ .

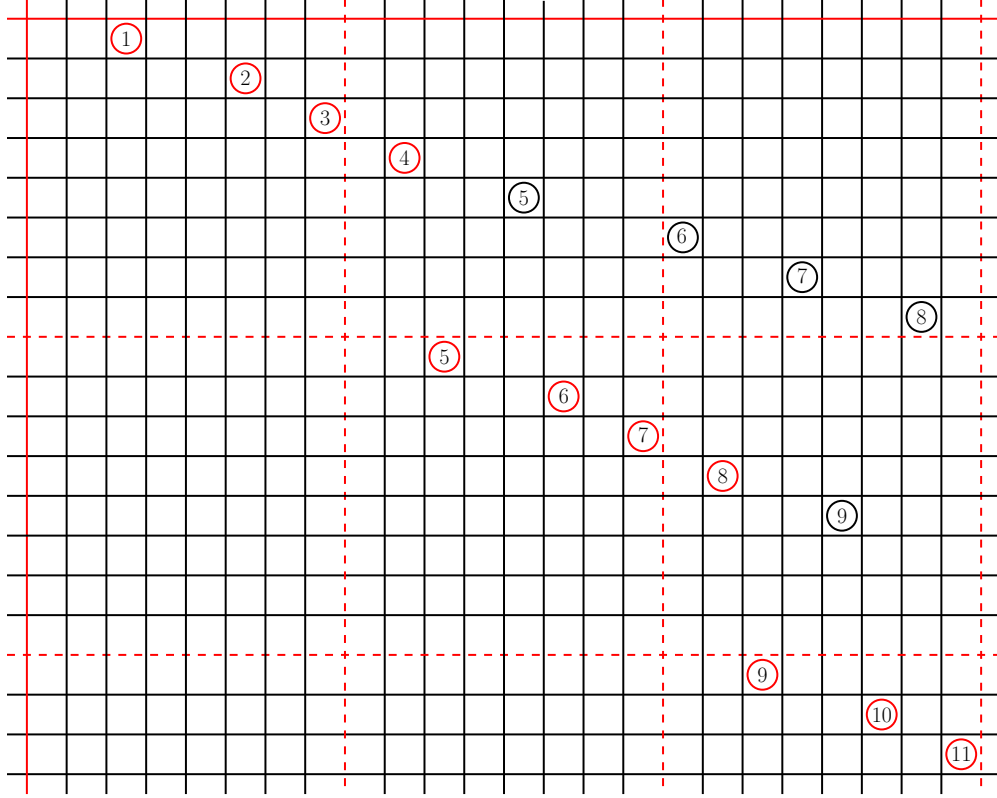


FIGURE 21. A combed permutation with southwest channel marked in red and with southwest channel numbering.

Consider the set of balls  $d^{-1}(i)$ . A ball in this set is specified by two parameters  $1 \leq j \leq n$  and  $k \in \mathbb{Z}$  satisfying one equation  $j + km = i$ ; its coordinates are  $(j + kn, w(j) + kn)$ . Consider two balls in this set with adjacent values of the parameter  $k$ . We will show that the ball  $b$  with parameters  $i - km$  and  $k$  is northeast of the ball  $b'$  with parameters  $i - (k + 1)m$  and  $k + 1$ . We have  $b = (i + k(n - m), w(i - km) + kn)$  and  $b' = (i + (k + 1)(n - m), w(i - (k + 1)m) + (k + 1)n)$ . It is clear that  $b'$  is south of  $b$  since  $n \geq m$ .

Now since  $i - (k + 1)m, i - km \in [n]$  and the permutation is combed, the balls in rows  $i - (k + 1)m, \dots, i - km$  form a chain with respect to  $\leq_{SE}$ . Since there are  $m + 1$  balls and all of them lie within  $n$  rows, they must not lie within  $n$  columns. Hence  $w(i - (k + 1)m) < w(i - km) - n$ . Thus  $w(i - (k + 1)m) + (k + 1)n < w(i - km) + kn$ . Hence  $b'$  is west of  $b$ , finishing the proof that  $d$  is a proper numbering.

It is clear that from every ball, one can take a path to the southwest channel so that the value of  $d$  decreases by one on each step. So  $d$  must be the southwest channel numbering.  $\square$

It is easy to see from the above lemmas that the first row of the tabloid resulting from Shi's algorithm coincides with the first row of the tabloid from AMBC. Moreover, one can see using the previous lemma that to get  $\text{fw}(w)$  from  $w$  one needs to erase the southwest channel and shift the remaining balls south by  $n - m$ . Since an overall shift south does not affect the  $P$ -tabloid, taking  $\text{fw}(w)$  corresponds to replacing a segment of the window in Shi's

algorithm with  $\emptyset$ 's. Where the  $\emptyset$ 's are in the permutation has no effect on the  $P$ -tabloid, so we may as well shift them in the initial rows. This completes the proof of Theorem 9.4.

### 13. A TALE OF TWO STREAMS

In this section we prove some basic results concerning backward numberings and use them to work out in detail what happens in the backward algorithm (i.e. when building  $w$  from  $(P, Q, \rho)$ ) when the shape of the tabloids  $P$  and  $Q$  is a two-row rectangle.

**13.1. Basic results concerning backward numberings.** First we prove that the backward numbering is well-defined.

*Proof of Proposition 4.3.* Consider a partial permutation  $w$  and a compatible stream  $S$ . We will first find a monotone numbering  $d' : \mathcal{B}_w \rightarrow \mathbb{Z}$  which is no larger than the stream numbering  $d_0$  on every ball. We will proceed to show that at every step of the backward numbering algorithm we preserve this property; namely for all  $b \in \mathcal{B}_w$  and all  $i$ ,  $d'(b) \leq d_i(b)$ . This shows that the algorithm necessarily terminates. We will then show that the result does not depend on the choices.

Let  $m_1$  be the width of the Shi poset of  $w$  and let  $m_2 \geq m_1$  be the flow of the stream  $S$ . Consider (any) proper numbering  $d_p$  of  $w$ . Define another numbering  $d'_p : \mathcal{B}_w \rightarrow \mathbb{Z}$  by

$$d'_p((i, w(i)) + k(n, n)) = d_p(i, w(i)) + km_2,$$

where  $1 \leq i \leq n$  and  $k \in \mathbb{Z}$ . It is easy to see  $d'_p$  is a monotone, semi-periodic numbering with period  $m_2$  (this uses the assumption that  $m_2 \geq m_1$ ). We know that  $d_0$  is a semi-periodic numbering with period  $m_2$ . So  $z := \max_{b \in \mathcal{B}_w} d'_p(b) - d_0(b)$  is well defined. Define  $d'$  by  $d'(b) = d'_p(b) - z$ . This is clearly a monotone, semi-periodic numbering of period  $m_2$  which is no larger than  $d_0$  on every ball.

Now suppose for some  $i$  and for all  $b \in \mathcal{B}_w$ ,  $d'(b) \leq d_i(b)$ . We will show that in this case, regardless of choice,  $d'(b) \leq d_{i+1}(b)$ . Choose a ball  $b$  such that there are balls southeast of it with the same value of  $d_i$ , but no such balls northwest of it. Let  $k = d_i(b)$ . We will consider  $d_{i+1}$  which results from decrementing the value on  $b$ . Choose  $b'$  southeast of  $b$  with  $d_i(b') = k$ . By assumption,  $d'(b') \leq k$ . But  $d'$  is monotone, so  $d'(b) < k$ . Thus  $d'(b) \leq k - 1 = d_{i+1}(b)$ . This finishes the proof that the backward numbering algorithm necessarily terminates.

The only thing remaining is to show that the result of the backward numbering algorithm is independent of the choices. This is done via a standard diamond lemma argument. Consider a numbering  $d_i$  which resulted from some choices. Suppose we have two possible numberings  $d_{i+1}$  and  $\tilde{d}_{i+1}$  which result from making different choices in the next step of the numbering algorithm. Suppose  $b$  and  $\tilde{b}$ , respectively, are the balls whose number had to be decremented to get the new numberings. Using the fact that  $d_i$  is weakly monotone, it is easy to see that  $\tilde{b}$  is a ball that can be chosen in the next step for  $d_{i+1}$  to produce  $d_{i+2}$ . Similarly,  $b$  is a ball that can be chosen in the next step for  $\tilde{d}_{i+1}$  to produce the same numbering  $d_{i+2}$ . Thus we have local confluence which implies global confluence by the diamond lemma.  $\square$

*Remark 13.1.* Suppose  $w$  is a partial permutation and  $S$  is a compatible stream. Suppose we have a monotone numbering  $d'$  which is nowhere greater than the stream numbering  $d_0$ . Then from the above proof it follows that  $d'$  is less than or equal to the backward numbering.



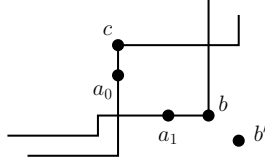


FIGURE 22. Cell positions in the proof of Lemma 13.4.

**Proposition 13.2.** *Suppose  $w$  is a partial permutation and  $S$  is a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Let  $d$  be the backward numbering of  $\mathcal{B}_w$ . Then any channel  $C$  of  $w$  is numbered by consecutive integers, i.e.  $d(C) = \mathbb{Z}$ .*

*Proof.* It is clear that  $d$  is semi-periodic with period equal to the flow of  $S$ . Consider  $b, b' \in C$  such that  $b' = b + (n, n)$ . Now following  $C$  from  $b$  to  $b'$ ,  $d$  must increase by at least 1 at every step. The number of steps is the width of the Shi poset of  $w$ . Since it is equal to the flow of  $S$ ,  $d$  must have increased by exactly 1 at each step. This finishes the proof.  $\square$

*Remark 13.3.* There must exist a ball whose numbering does not change during the algorithm; i.e. for which the backward numbering and the stream numbering match. Otherwise we could add 1 to the backward numbering of all balls and get a monotone numbering which is nowhere greater than  $d_0$ . This contradicts Remark 13.1.

A step of the backward algorithm produces a partial permutation, and this permutation comes with an induced numbering; we will now show that this numbering is proper.

**Lemma 13.4.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . A step of the backward algorithm involves a collection of reverse zig-zags. These zig-zags do not intersect; moreover every cell of the zig-zag labeled  $i$  is strictly southeast of some cell of the zig-zag labeled  $i - 1$ .*

*Proof.* By construction, we know that the southwest (resp. northeast) ball of  $\text{bk}_S(w)$  labeled  $i$  is east (resp. south) of the southwest (resp. northeast) ball of  $\text{bk}_S(w)$  labeled  $i - 1$ . So, any cell of the  $i$ -th zig-zag is either southeast or northwest of some cell of the  $(i - 1)$ -st zig-zag.

Suppose there exists a cell  $a_0$  of the  $i$ -th zig-zag northwest of a cell  $a_1$  of the  $(i - 1)$ -st zig-zag (see Figure 22). Choose a ball  $c$  of  $\text{bk}_S(w)$  on the  $i$ -th zig-zag either directly north or directly west of  $a_0$ . Then  $c$  is northwest of  $a_1$ . Choose a ball  $b$  of  $w$  on the  $(i - 1)$ -st zig-zag either directly south or directly east of  $a_1$ . So  $c$  is northwest of  $b$ . But then  $b$  is not labeled by highest element of  $S$  northwest of it, so  $b$  was bumped during the backward numbering algorithm. Hence there exists a ball  $b'$  of  $w$  labeled  $i$  southeast of  $b$  and hence strictly southeast of  $c$ . This is of course impossible since  $b'$  and  $c$  must lie on the same zig-zag. This is a contradiction.  $\square$

**Lemma 13.5.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . The step of the backward algorithm induces a numbering and a partition into zig-zags for  $\text{bk}_S(w)$ ; call the numbering  $d$ . Then  $d$  is always a proper numbering.*

*Proof.* Any balls of  $\text{bk}_S(w)$  comparable in the northwest ordering must lie in different zig-zags. The conditions of the previous lemma, imply that in this case the northwest of these zig-zags has strictly lower number. Thus  $d$  is a monotone numbering. Hence the only thing

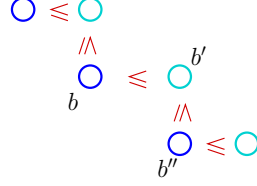


FIGURE 23. Inequalities for backward numberings of  $t$  and  $t'$ . The balls of  $t$  are dark blue while those for  $t'$  are light blue.

we need to check is that for any ball  $c$  of  $\text{bk}_S(w)$  with  $d(c) = i$ , there exists a ball  $c'$  northwest of  $c$  with  $d(c') = i - 1$ . This also follows from the previous lemma: we know that there exists some cell  $a$  of the  $(i - 1)$ -st zig-zag northwest of  $c$  and we can choose  $c'$  to be a ball of  $\text{bk}_S(w)$  either directly north or directly west of  $a$ .  $\square$

*Remark 13.6.* We will later see (Proposition 14.19) that the numbering in the above lemma is a channel numbering.

As a corollary we have:

**Corollary 13.7.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . Then the width of the Shi poset of  $\text{bk}_S(w)$  is equal to the flow of  $S$ .*

*Proof.* It is clear that the period of the induced numbering is equal to the flow of  $S$ . By the previous lemma, Proposition 3.4 applies to finish the proof.  $\square$

**13.2. The case of two-row rectangles.** In this section we explore in detail the notion of stream concurrency; equivalently we study what happens when the Greene-Kleitman shape of the Shi poset is a two-row rectangle. We will generally consider two streams  $S$  and  $T$  of the same flow. We will view the cells of  $T$  as a balls of a partial permutation  $t = \text{bk}_T([\emptyset, \dots, \emptyset])$ .

We begin by introducing the operations of shifting streams and channels.

**Definition 13.8.** Suppose  $S = \text{st}_r(A, B)$  is a stream. Then the shift of  $S$  by  $k$  is the stream

$$S \langle k \rangle := \text{st}_{r+k}(A, B).$$

**Definition 13.9.** Suppose  $t$  is a permutation consisting of a single channel; say  $t = \{(p_i, q_i) : i \in \mathbb{Z}\}$  where for each  $i$ ,  $(p_i, q_i)$  is northwest of  $(p_{i+1}, q_{i+1})$ . Then the shift of  $t$  by  $k$  is the permutation  $t \langle k \rangle = \{(p_i, q_{i+k}) : i \in \mathbb{Z}\}$

**Lemma 13.10.** *Consider a partial permutation  $t$  consisting of a single channel and a compatible stream  $S$  whose flow is equal to the width of the Shi poset of  $t$ . Let  $t' = t \langle 1 \rangle$ . Let  $d$  (resp.  $d'$ ) be the backward numbering of  $t$  (resp.  $t'$ ) with respect to  $S$ . Then for any ball  $b' \in \mathcal{B}_{t'}$ , we have  $d(b) \leq d'(b') \leq d(b'')$ , where  $b \in t$  is the ball directly west of  $b'$  and  $b''$  is the ball directly south of  $b'$  (see Figure 23).*

*Proof.* Let  $d_0$  (resp.  $d'_0$ ) be the stream numbering of  $t$  (resp.  $t'$ ). Denote by  $d_i$  (resp.  $d'_i$ ) the numbering on step  $i$  of the backward algorithm. It is clear that for any triple  $b, b', b''$  as in the statement, we have  $d_0(b) \leq d'_0(b') \leq d_0(b'')$ . We will show that (given suitable choices)  $d_i(b) \leq d'_i(b') \leq d_i(b'')$ .

Assume not; suppose the first failure is that at some point  $d'_i(b') < d_i(b)$  (the argument if the other inequality fails is similar). Let  $b'''$  be the ball of  $t'$  directly east of  $b''$ . Then just before the decrement we have  $d'_{i-1}(b') = d_{i-1}(b)$  and  $d'_{i-1}(b') = d'_{i-1}(b''')$ . Now the inequalities held on step  $i-1$ , so  $d_{i-1}(b'') = d'_{i-1}(b')$ . Similarly,  $b$  was the northwest ball of  $t$  with this value of  $d_{i-1}$ , since  $b'$  was the northwest ball of  $t'$  with this value of  $d'_{i-1}$ . Thus we could have made a choice to decrement the value of  $b$  to form  $d_i$ ; with this choice we would have  $d_i(b) \leq d'_i(b') \leq d_i(b'')$ .  $\square$

Thus the backward numbering of  $t$  matches that of  $t\langle 1 \rangle$  on either rows or columns. The next result shows that once the numbering starts matching on rows, it will keep doing so as we shift  $t$  further.

**Lemma 13.11.** *Consider a partial permutation  $t$  consisting of a single channel and a compatible stream  $S$  whose flow is equal to the width of the Shi poset of  $t$ . Let  $t' = t\langle 1 \rangle$ , and  $t'' = t'\langle 1 \rangle = t\langle 2 \rangle$ . Let  $d$  be the backward numbering of  $t$  with respect to  $S$ , and similarly  $d'$  and  $d''$ . Suppose that for some (equiv. any) ball  $b \in t$ , we have  $d(b) = d'(b')$ , where  $b'$  is the ball of  $t'$  directly east of  $b$ . Then  $d'(b') = d''(b'')$ , where  $b''$  is the ball of  $t''$  directly east of  $b'$ .*

*Proof.* Recall the notation  $S^{(i)}$  for the ball of  $S$  labeled  $i$ . Choose  $b$  so that  $b'$  has backward numbering equal to stream numbering (as in Remark 13.3). Let  $i = d'(b')$ . Now the ball of  $t$  directly south of  $b'$  is numbered  $i+1$ , so  $S^{(i+1)}$  must be southwest of  $b'$ . Hence  $S^{(i+1)}$  is southwest of  $b''$ . Thus  $d''(b'') \leq i$ . By the previous lemma, however  $i \leq d''(b'') \leq i+1$ . Hence  $d''(b'') = i$ , as desired.  $\square$

*Remark 13.12.* The above lemma can be reflected in the main diagonal. Let  $t$  and  $S$  be as above; let  $t' = t\langle -1 \rangle$  and  $t'' = t\langle -2 \rangle$ . Suppose that the backward numbering of a ball of  $t$  matches the backward numbering of a ball of  $t'$  directly south of it. Then the backward numbering of a ball of  $t'$  matches the backward numbering of a ball of  $t''$  directly south of it.

Consider a collection of streams  $\mathbf{st}(A, B)$ ; its members are  $\mathbf{st}_r(A, B)$  for  $r \in \mathbb{Z}$ . For  $r \in \mathbb{Z}$  let  $t_r$  be the partial permutation with the same cells as  $\mathbf{st}_r(A, B)$ . From the above lemma and its reflection we conclude that there exists a unique  $r_0$  such that for any  $k > 0$  the backward numberings of  $t_{r_0}$  and  $t_{r_0+k}$  coincide on rows while the backward numberings of  $t_{r_0}$  and  $t_{r_0-k}$  coincide on columns.

This allows us to extend the notion of altitude to streams of the same flow but in different classes.

**Definition 13.13.** Suppose  $S \in \mathbf{st}(A, B)$  and  $T \in \mathbf{st}(A', B')$  are streams of the same flow and  $A \cap A' = \emptyset$ ,  $B \cap B' = \emptyset$ . Then  $T$  is *no lower than*  $S$  if  $T = \mathbf{st}_r(A', B')$  for  $r \geq r_0$ , where  $r_0$  was defined in the above remark. Similarly, we can define when  $T$  is *no higher than*  $S$ . If  $T = \mathbf{st}_{r_0}(A', B')$ , we say that  $T$  is *at the same height as*  $S$ .

In the next series of results we prove that the notion of  $T$  having the same height as  $S$  is precisely the notion of  $T$  being concurrent to  $S$  from Definition 5.4. In particular there is the same terminological problem as with the definition of concurrency; namely that the definition is not symmetric and it can happen that  $S$  is at the same height as  $T$  but  $T$  is not at the same height as  $S$ . We begin by proving a lemma necessary to make sense of the definition of concurrency, namely that the result of a backward step in our case partitions into two channels.

*Proof of Lemma 5.3.* We know the width of the Shi poset of  $\text{bk}_S(t)$  by Corollary 13.7. The backward numbering assigns consecutive integers to elements of  $t$ , so the Young diagrams in the backward step are all rectangles. It is easy to see that their southwest corners form a channel and that their northeast corners form a channel.  $\square$

**Lemma 13.14.** *Let  $T$  be a stream with corresponding permutation  $t$ . Consider a compatible stream  $S$  whose flow is equal to the width of the Shi poset of  $t$ . Let  $\tilde{d}$  be the proper numbering induced by the backward step on  $\text{bk}_S(t)$ . Then  $\tilde{d}$  is the southwest channel numbering if and only if the  $T$  is no lower than  $S$ .*

*Proof.* We know that  $\text{bk}_S(t)$  consists of a southwest channel, call it  $C_1$ , and a northeast channel,  $C_2$ . We know that a proper numbering is the southwest numbering precisely when there exists a ball  $b$  of  $C_2$  such that the ball of  $C_1$  whose number is one less lies northwest of  $b$ . As usual, we let  $t' = t \langle 1 \rangle$ . Let  $d$  (resp.  $d'$ ) be the backward numbering of  $t$  (resp.  $t'$ ) with respect to  $S$ . We will only show the forward implication since the backward one involves the same steps in reverse order.

Suppose  $T$  is no lower than  $S$ . We know that the backward numberings of  $t$  and  $t'$  match on rows. Let  $b'$  be the ball of  $t'$  for which the backward and stream numberings coincide; let  $i = d'(b')$ . Then  $S^{(i+1)}$  lies south of  $b'$  (it cannot lie northwest of  $b'$  and must lie northwest of the ball of  $t$  directly south of  $b'$ ). The ball of  $C_1$  labeled  $i$  (by  $\tilde{d}$ ) lies directly west of  $b'$ . The ball of  $C_2$  labeled  $i + 1$  lies directly east of  $S^{(i+1)}$  and east of  $b'$ . This is what was necessary to show that  $\tilde{d}$  is the southwest channel numbering.  $\square$

Now we are ready to show that the notions of having the same height and concurrency are, in fact, the same. Moreover, we will conclude that concurrency is preserved by shifting the streams simultaneously. These statements comprise the main result in Section 5.1.

*Proof of Proposition 5.6.* Recall the notation; for sets  $A, B, A', B'$  of the same size, we need to show that there exists a unique  $r$  such that  $T := \mathfrak{st}_r(A', B')$  is concurrent to  $S := \mathfrak{st}_0(A, B)$ . This follows easily, with  $r = r_0$  as above, from the previous lemma and its reflection in the main diagonal.

Let us now show that  $T' := \mathfrak{st}_{r+1}(A', B')$  is concurrent to  $S' := \mathfrak{st}_1(A, B)$ ; repeating the argument and reflecting about the main diagonal will finish the proof. Let  $t$  (resp.  $t'$ ) be the partial permutation corresponding to  $T$  (resp.  $T'$ ). Consider  $S'$  with proper numbering which matches that of  $S$  on rows. Let  $d$  be the backward numbering of  $t$  with respect to  $S$  and let  $d'$  be the numbering of  $t'$  which coincides with  $d$  on rows. We will now show that  $d'$  is the backward numbering of  $t'$  with respect to  $S'$ .

It is easy to see, using Remark 13.1, that  $d'$  is bounded above by the backward numbering. Also there exists a ball  $b'$  of  $t'$  such that  $S'^{(d'(b')+1)}$  lies south of  $b'$  (consider the ball of  $t'$  whose backward numbering with respect to  $S$  is equal to the stream numbering, as in Lemma 13.11). We must have  $d'(b')$  coinciding with the backward numbering of  $b'$ . Hence the two numberings must always coincide.

As noted above,  $b'$  lies north of  $S'^{(d'(b')+1)}$ , so  $T'$  is no lower than  $S'$ . Similarly, let  $b \in t$  be the ball such that  $S^{(d(b)+1)}$  is east of  $b$ . Then the ball  $b''$  of  $t'$  directly north of  $b$  has  $d'(b'') = d(b) - 1$  and  $S'^{(d(b))}$  is east of it. Hence  $T'$  is indeed concurrent to  $S'$ , finishing the proof.  $\square$

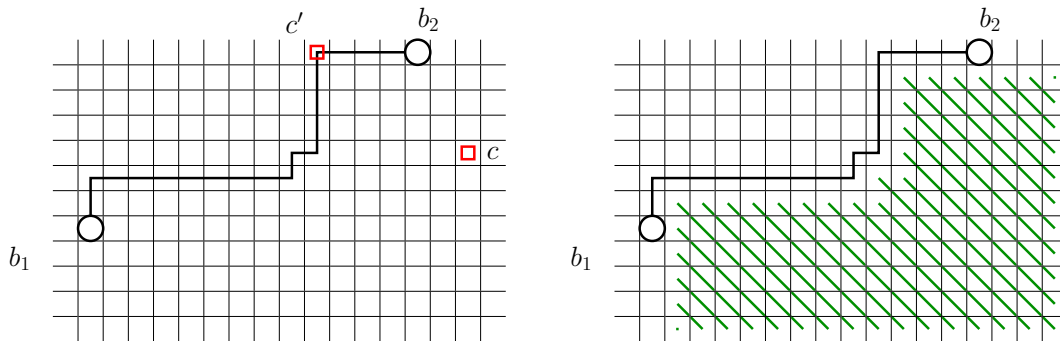


FIGURE 24. Cell positioning in Lemma 14.1.

## 14. WEIGHTS

**14.1. A key lemma.** In this section we will present a key (trivial) lemma, and then give examples of constructions resulting from it and several applications. The constructions will be used over and over again in the rest of the paper.

**Lemma 14.1.** *Consider a cell  $c$  and two cells  $b_1, b_2$  such that  $b_1$  is strictly east (and possibly south or north) of  $c$ ,  $b_2$  is strictly north (and possibly east or west) of  $c$ , and  $b_1$  is southwest of  $b_2$  (see the left side of Figure 24). Consider a zig-zag (recall Definition 3.35) from  $b_1$  to  $b_2$  which has no cell southeast of  $c$ . Then there exists a northwest corner  $c'$  of the zig-zag (a north step followed by an east step) such that  $c'$  is northwest of  $c$ .*

*Proof.* It is clear that for every cell  $c$  in the region shaded in the right half of Figure 24 one can find such  $c'$ . The conditions on  $b_1, b_2$  and the zig-zag ensure that  $c$  lies in the shaded region.  $\square$

Of course, there is also a version of this lemma obtained by reflection in an anti-diagonal.

Although the lemma is not difficult, its repeated application allows us to make several constructions which will be the main ingredients of many proofs in the rest of the paper.

*Example 14.2.* Consider a partial permutation  $w$  and the reverse zig-zags defined by a proper numbering  $d$ . Suppose we have two paths on the set of balls of  $\text{fw}_d(w)$ :  $p = (b_0, b_1, \dots, b_k)$  and  $q = (c_0, c_1, \dots, c_k)$  (recall that a path is a sequence of balls where each ball is northwest of the previous one). Moreover assume that for all  $i$ ,  $b_i$  lies southwest of  $c_i$  on the same zig-zag, and that  $b_k = c_k$ . Then starting with any ball  $a_0$  of  $w$  between  $b_0$  and  $c_0$  on the same zig-zag, one can construct a path  $(a_0, a_1, \dots, a_{k-1})$  such that each  $a_i$  is between  $b_i$  and  $c_i$  in each zig-zag. The path constructed in this way will be referred to as a *funnel walk*; an example is shown in the left hand side of Figure 25. A *reverse funnel walk*, obtained via reflection in an anti-diagonal, is shown in the right hand side of the same figure.

*Example 14.3.* Consider a permutation  $w$  and the reverse zig-zags defined by a proper numbering  $d$ . Consider two infinite paths  $p = (b_0, b_1, \dots)$  and  $q = (c_0, c_1, \dots)$  of  $\text{fw}_d(w)$  such that for each  $i$ ,  $b_i$  is southwest of  $c_i$  on the same zig-zag. Start with any point  $a_0 \in \mathcal{B}_w$  which is between  $b_0$  and  $c_0$  on their common zig-zag. By repeated application of the lemma one finds an infinite path  $(a_0, a_1, \dots)$  such that each  $a_i$  lies between  $b_i$  and  $c_i$  on its zig-zag. The

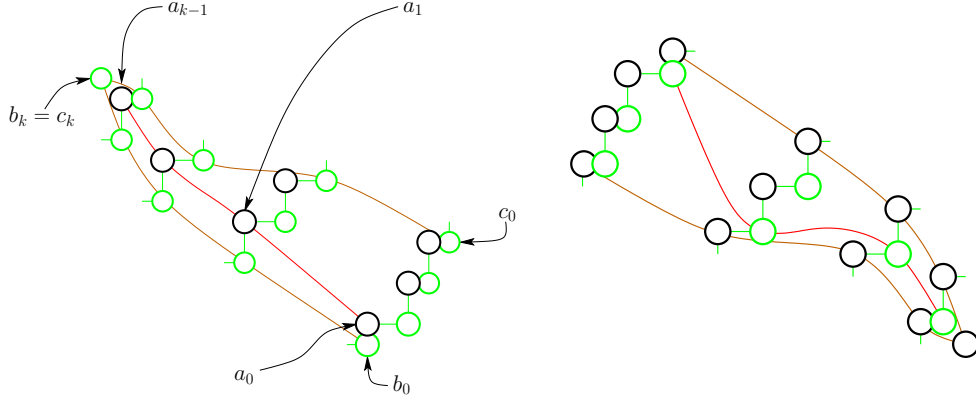


FIGURE 25. A funnel walk and a reverse funnel walk. The balls of  $w$  are shown in black; the balls of  $\text{fw}(w)$  are shown in green. The paths  $p$  and  $q$  are shown with brown curves. The walk  $(a_0, a_1, \dots, a_{k-1})$  is indicated by the red line.

path constructed in this way will be referred to as a *bounded walk*. A *bounded reverse walk* is obtained via reflection in an anti-diagonal.

The construction below is similar in nature, but does not use the lemma directly.

*Example 14.4.* Consider a partial permutation  $w$  and the reverse zig-zags defined by a proper numbering  $d$ . Consider a (finite or infinite) path  $p = (b_0, b_1, \dots)$ . Choose some  $a_0 \in \mathcal{B}_w$  on the zig-zag of  $b_0$ . Without loss of generality,  $a_0$  is northeast of  $b_0$ . Since the numbering was proper, there exist balls of  $w$  northwest of  $a_0$  in the zig-zag of  $b_1$ . We show that at least one of them is northeast of  $b_1$ . The ball  $b_1$  is west of  $b_0$  and hence west of  $a_0$ . If  $b_1$  is north of  $a_0$  then the ball of  $w$  directly north of  $b_1$  is also northwest of  $a_0$ . If  $b_1$  is south of  $a_0$  then any ball of its zig-zag which is northwest of  $a_0$  is also northeast of  $b_1$ . Repeating the argument gives a path  $(a_0, a_1, \dots)$  northeast of  $p$  and having the same length. We will refer to such a path as a *semi-bounded walk*. An example is shown in the left side of Figure 26.

Unlike with the other two constructions, a *semi-bounded reverse walk* does not quite behave symmetrically (due to the fact that the ends of the zig-zags are always balls of  $w$ , not  $\text{fw}_d(w)$ ). The difference is that the reverse walk sometimes is forced to stop; it is not always possible to find the next element.

Consider a (finite or infinite) reverse path  $p = (c_0, c_1, \dots)$ . Choose some  $a_0 \in \mathcal{B}_{\text{fw}_d(w)}$  on the zig-zag of  $c_0$ . Without loss of generality,  $a_0$  is northeast of  $c_0$ . The argument from the previous paragraph can fail in two ways. First, there do not have to be balls of  $\text{fw}_d(w)$  in the zig-zag of  $c_1$  southeast of  $a_0$ . Second, even if there is such a ball, we cannot repeat the argument since there may not be a ball of  $\text{fw}_d(w)$  directly east of  $c_1$ . Thus (in either case) if we could not find the next element  $a_1$  in a semi-bounded reverse walk, then the zig-zag containing  $c_1$  lies completely south of  $a_0$ . An example is shown in the right of Figure 26.

**14.2. Applications to channels.** In this section we apply the above constructions to prove some basic results about channels. The first several results describe the relative position of channels of a partial permutation  $w$  and the partial permutations  $\text{fw}_d(w)$  and  $\text{bk}_S(w)$ .

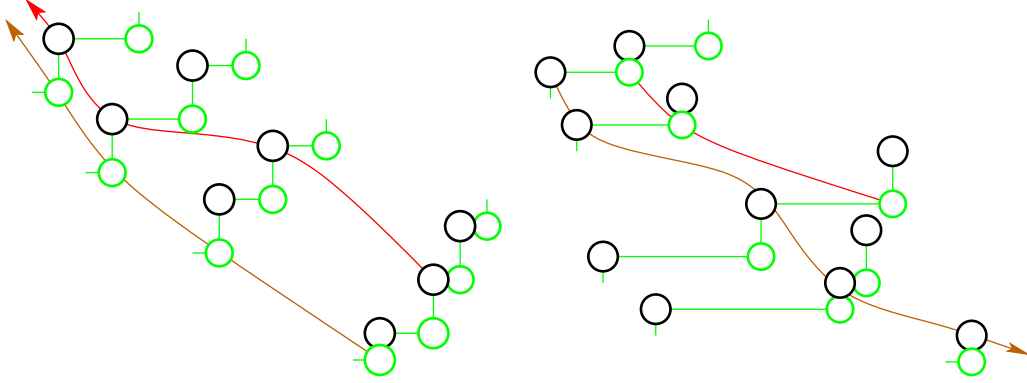


FIGURE 26. A semi-bounded walk and a semi-bounded reverse walk (the latter is sometimes forced to stop).

**Proposition 14.5.** *Consider a partial permutation  $w$  with a proper numbering  $d$  with its reverse zig-zags. Suppose we have two disjoint channels  $C_1, C_2 \in \mathcal{C}_w$  and  $C_1$  is southwest of  $C_2$ . Then there exists a channel  $D \in \mathcal{C}_{\text{fw}_d(w)}$  such that each ball of  $D$  is located between the ball of  $C_1$  and the ball of  $C_2$  on its zig-zag. Also, for any channel  $D$  of  $\text{fw}_d(w)$  there exist a channel  $C_1$  of  $w$  southwest of it and a channel  $C_2$  northeast of it.*

*Proof.* For each  $i$ , let  $b_i \in C_1$  (resp.  $c_i \in C_2$ ) be the ball of  $C_1$  (resp.  $C_2$ ) in the  $i$ -th zig-zag. Let  $a_0$  be a ball of  $\text{fw}(w)$  on the 0-th zig-zag which lies between  $b_0$  and  $c_0$ . Let  $(a_0, a_1, \dots)$  be the corresponding bounded reverse walk.

By pigeonhole principle, this reverse walk contains two  $(n, n)$ -translates of each other. Up to reindexing, we can assume that for some  $l$ ,  $a_0$  and  $a_{l-1}$  are  $(n, n)$ -translates. Consider the partial permutation  $w'$  whose balls are  $\{b_0, \dots, b_{l-1}\}$  as well as all their  $(n, n)$ -translates. The restriction of  $d$  to the balls of  $w'$  is obviously a proper numbering, hence it must be semi-periodic. Since the period is equal to the width of the Shi poset, the Shi poset of  $w'$  must contain a maximal antichain of the Shi poset of  $\text{fw}(w)$ . The channel corresponding to this maximal antichain is the sought channel  $D$ .

To prove the last statement we start with a ball southwest of  $D$  and a ball northeast of  $D$ . The semi-bounded (by  $D$ ) walks starting at these balls arrive at channels  $C_1$  and  $C_2$  by the same argument as before.  $\square$

**Corollary 14.6.** *Suppose  $w$  is a partial permutation whose Shi poset has  $k \geq 2$  disjoint maximal antichains; let  $d$  be a proper numbering. Let  $\{C_1, \dots, C_k\}$  be the largest disjoint collection of channels such that  $C_i$  is southwest of  $C_{i+1}$  for all  $i$ . For each  $i$  choose a channel  $D_i$  of  $\text{fw}_d(w)$  between  $C_i$  and  $C_{i+1}$  as done in Proposition 14.5. Then the channels  $D_1, \dots, D_{k-1}$  form a largest disjoint collection of channels of  $\text{fw}_d(w)$ .*

*Proof.* It is clear that the channels  $D_1, \dots, D_{k-1}$  are pairwise disjoint. If one could arrange more than  $k - 1$  such channels, then by the above proposition, we would arrange more than  $k$  disjoint channels for  $w$ .  $\square$

The next result is a similar statement for a step of the backward algorithm.

We know by Corollary 13.7 that the width of the Shi poset of  $\text{bk}_S(w)$  is equal to the flow of  $S$ .

**Proposition 14.7.** *Suppose  $w$  is a partial permutation and  $S$  is a compatible stream. Moreover assume that the width of the Shi poset of  $w$  is equal to the flow of  $S$ . Choose a maximal disjoint collection of channels  $\{D_1, \dots, D_{k-1}\}$  such that for any  $i$ ,  $D_i$  is southwest of  $D_{i+1}$ . The induced numbering on  $\text{bk}_S(w)$  is proper (by Lemma 13.5); consider the zig-zags defined by that numbering. Then there exists a channel  $C_1$  of  $\text{bk}_S(w)$  whose cells are southwest of the cells of  $D_1$ , a channel  $C_k$  whose cells are northeast of the cells of  $D_{k-1}$ , and for each  $2 \leq i \leq k-1$  a channel  $C_i$  between  $D_{i-1}$  and  $D_i$ . The channels  $C_1, \dots, C_k$  form a maximal disjoint collection of channels of  $\text{bk}_S(w)$ .*

*Proof.* There exists a ball of  $\text{bk}_S(w)$  directly west of a ball of  $D_1$ . A semi-bounded walk starting at this element with respect to  $D_1$  must be infinite as described in Example 14.4. By the same argument as in Proposition 14.5, this walk comes to a channel. This proves the existence of  $C_1$ . Analogously we can conclude the existence of  $C_2, \dots, C_k$ . It is clear that these form a maximal collection of channels of  $\text{bk}_S(w)$  since between any two channels of  $\text{bk}_S(w)$  there is a channel of  $w$ .  $\square$

Now let us describe when certain walks can be chosen to intersect channels. In the proof of Proposition 14.5, we saw that a bounded walk (resp. bounded reverse walk) between two channels of  $\text{fw}_d(w)$  (resp. of  $w$ ) necessarily intersects some channel of  $w$  (resp.  $\text{fw}_d(w)$ ). Similarly an infinite semi-bounded walk (or infinite semi-bounded reverse walk) always reaches a channel.

*Remark 14.8.* Suppose  $w$  is a partial permutation whose Shi poset has  $k \geq 2$  disjoint maximal antichains. Let  $(C_1, \dots, C_k)$  and  $(D_1, \dots, D_{k-1})$  be interlacing collection of channels as in Corollary 14.6. We know that for any  $i$ , a bounded walk between  $D_i$  and  $D_{i+1}$  necessarily intersects some channel  $C$  (which lies completely between  $D_i$  and  $D_{i+1}$ ). However  $C$  must intersect  $C_{i+1}$  since otherwise there would be too many disjoint channels of  $w$ . Hence one can continue the walk by following  $C$  until reaching  $C_{i+1}$ . Thus from any ball between  $D_i$  and  $D_{i+1}$  we can choose a bounded walk which intersects our chosen channel  $C_{i+1}$ . Exactly the same story holds for bounded reverse walks. Also, starting with a ball of  $\text{fw}(w)$  southwest (resp. northeast) of  $D_1$  (resp.  $D_{k-1}$ ), a semi-bounded walk can be chosen to intersect  $C_1$  (resp.  $C_k$ ).

**14.3. Dominance in the image of  $\Phi$ .** The main goal of this section is to prove that the image of  $\Phi$  is subset of  $\Omega_{\text{dom}}$ . Before getting to the proof, we study two monotone numberings of the balls of  $\text{fw}(w)$ . One is its own southwest channel numbering  $d_{\text{fw}(w)}^{\text{SW}}$ . The second one is the numbering afforded by the zig-zags from  $d_w^{\text{SW}}$  as follows.

**Definition 14.9.** Suppose  $d$  is a proper numbering of  $w$ . Define  $\text{fw}(d) : \mathcal{B}_{\text{fw}_d(w)} \rightarrow \mathbb{Z}$  by  $(\text{fw}(d))(b) = i$  if  $b$  is part of the  $i$ -th zig-zag of  $d$ .

*Remark 14.10.* The two numberings  $d_{\text{fw}(w)}^{\text{SW}}$  and  $\text{fw}(d_w^{\text{SW}})$  do not always coincide; moreover the numbering  $\text{fw}(d_w^{\text{SW}})$  is not always proper.

The next result demonstrates that, although the two numberings do not coincide everywhere, they (can be shifted to) coincide on channels.

**Proposition 14.11.** *Consider a partial permutation  $w$  whose Shi poset has at least two disjoint maximal antichains. Fix some shift of  $d_w^{\text{SW}}$  and choose the shift of  $d_{\text{fw}(w)}^{\text{SW}}$  to coincide*



with  $\text{fw}(d_w^{SW})$  on the southwest channel of  $\text{fw}(w)$ . Then the two numberings coincide on all the channels of  $\text{fw}(w)$ . Moreover, for any  $b \in \mathcal{B}_{\text{fw}(w)}$  we have  $(\text{fw}(d_w^{SW}))(b) \geq d_{\text{fw}(w)}^{SW}(b)$ .

*Proof.* Consider interlacing maximal collections of channels  $\{C_1, \dots, C_k\}$  of  $w$  and  $\{D_1, \dots, D_{k-1}\}$  of  $\text{fw}(w)$  (as described in Corollary 14.6). It is sufficient to show that  $(\text{fw}(d_w^{SW}))(b) = d_{\text{fw}(w)}^{SW}(b)$  for any ball  $b$  in  $D_1 \cup \dots \cup D_{k-1}$  since any channel of  $\text{fw}(w)$  intersects one of these and both  $\text{fw}(d_w^{SW})$  and  $d_{\text{fw}(w)}^{SW}$  number the channels by consecutive integers.

We know that the numbering  $\text{fw}(d_w^{SW})$  is monotone and that  $D_1$  intersects the southwest channel of  $\text{fw}(w)$ . By the Remark 11.7, to check whether  $\text{fw}(d_w^{SW})$  coincides with  $d_{\text{fw}(w)}^{SW}$  on a certain ball  $b$ , it is sufficient to check that there is a path  $(b = b_0, b_1, \dots, b_l)$  to  $D_1$  such that  $\text{fw}(d_w^{SW})(b_{i+1}) = \text{fw}(d_w^{SW})(b_i) - 1$ . It remains to construct the desired path from a ball  $b$  in any  $D_i$ . For every  $i$ , we will construct a such a path from  $D_i$  to  $D_{i-1}$ . We can then get from  $D_{k-1}$  to  $D_1$  as follows. Take the path from  $D_{k-1}$  to  $D_{k-2}$ , then follow  $D_{k-2}$  to the start of the path to  $D_{k-3}$ . Take the path to  $D_{k-3}$  and follow  $D_{k-3}$  until the start of the path to  $D_{k-4}$ . Repeat this procedure until arriving at  $D_1$ .

It is easier to construct the path in reverse; i.e. from  $D_{i-1}$  to  $D_i$ . The construction of the reverse path will proceed in two parts. The first part will use a reverse funnel walk whose purpose is to get a reverse path to some point above  $C_{i-1}$  (it is illustrated in Figure 27). The second part is a bounded reverse walk from a ball of  $\text{fw}(w)$  between  $C_{i-1}$  and  $C_i$  to  $D_i$ .

Consider a path  $p = (c_0, c_1, \dots, c_h)$  from  $C_{i-1}$  to  $C_{i-2}$  such that  $d_w^{SW}$  decreases by 1 at each step. Start with the ball  $b'_0 \in D_{i-1}$  which lies on the same zig-zag as  $c_h$  and is located between  $C_{i-1}$  and  $C_{i-2}$ . Let  $(b'_0, \dots, b'_{h-1})$  be the funnel walk between  $p$  and  $C_{i-1}$ . Let  $b'_h$  be the ball of  $\text{fw}(w)$  directly east of  $c_0$ . Let  $(b'_h, b'_{h+1}, \dots, b'_{h'})$  be a reverse bounded walk between  $C_{i-1}$  and  $C_i$  from  $b'_h$  to  $D_i$  (as described in Remark 14.8). Thus we have constructed a path with the desired properties, finishing the proof that the two numberings coincide on channels.

Observe that at each step of a path of maximal worth from any ball of  $\text{fw}(w)$  to the southwest channel,  $d_{\text{fw}(w)}^{SW}$  increases by 1 while  $\text{fw}(d_w^{SW})$  increases by at least 1. This demonstrates the last claim.  $\square$

We proceed to a technical lemma, which handles the crux of the proof that the image of  $\Phi$  is a subset of  $\Omega_{\text{dom}}$ . Recall that whenever we have a stream  $S$  and a proper numbering of it, then  $S^{(i)}$  refers to the cell of the stream numbered  $i$ .

**Lemma 14.12.** *Let  $w$  be a partial permutation whose Shi poset has  $k \geq 2$  maximal antichains. Let  $S = \mathbf{st}(w)$ , and let  $T = \mathbf{st}(\text{fw}(w))$ . Consider some fixed numbering  $d_w^{SW}$  and the numbering  $d_{\text{fw}(w)}^{SW}$  which coincides with  $\text{fw}(d_w^{SW})$  on channels. The stream  $S$  (resp.  $T$ ) has a natural numbering  $d_S : S \rightarrow \mathbb{Z}$  (resp.  $d_T : T \rightarrow \mathbb{Z}$ ) coming from  $d_w^{SW}$  (resp.  $d_{\text{fw}(w)}^{SW}$ ). In this case for some  $i$ , we have  $T^{(i)}$  lies north  $S^{(i+1)}$ .*

*Proof.* Choose interlacing collections of channels  $\{C_1, \dots, C_k\}$ , and  $\{D_1, \dots, D_{k-1}\}$  as before. Consider a path  $p = (c_0, \dots, c_l)$  from  $C_k$  to  $C_1$  such that  $d_w^{SW}$  decreases by 1 at each step. Start with the element  $b_0$  of  $D_1$  on the same zig-zag as  $c_l$ . Construct a funnel reverse walk  $(b_0, \dots, b_{l-1})$  between  $p$  and  $C_k$ . Notice that  $\text{fw}(d_w^{SW})$  and  $d_{\text{fw}(w)}^{SW}$  coincide on  $b_0$  since it is

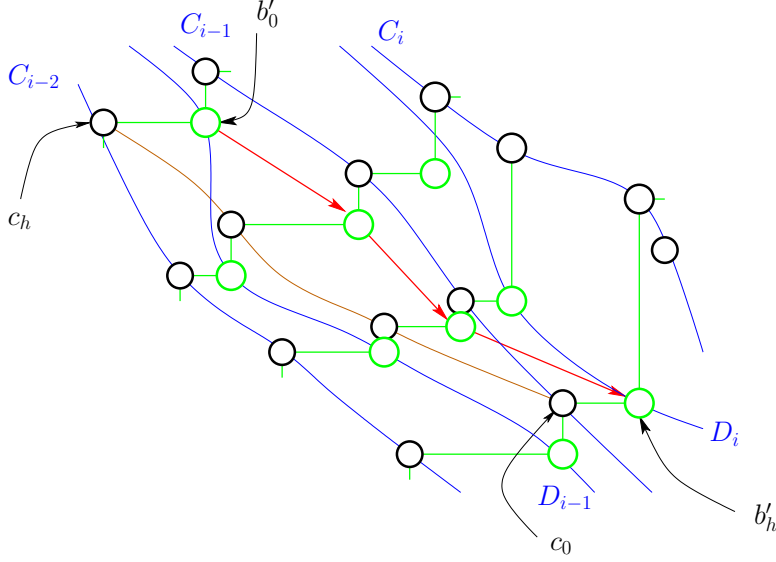


FIGURE 27. A reverse path from  $D_{i-1}$  to  $D_i$ . The balls of  $w$  are in black, while the balls of  $\text{fw}(w)$  are in green. The channels are indicated by thin blue lines. The path from  $C_{i-1}$  to  $C_{i-2}$  is indicated by the brown line.

part of a channel, and hence must coincide on all of  $b_1, \dots, b_{l-1}$  since  $\text{fw}(d_w^{SW})$  increases by one on each step while  $d_{\text{fw}(w)}^{SW}$  is bounded above by it.

Let  $i$  be the label of  $b_{l-1}$  (it is the same in both relevant numberings of  $\mathcal{B}_{\text{fw}(w)}$ ). Suppose the  $(i+1)$ -st zig-zag does not extend north of  $b_{l-1}$ . In this case we are done since  $T^{(i)}$  is north of  $b_{l-1}$  while  $S^{(i+1)}$  is south of  $b_{l-1}$ . Now assume the  $(i+1)$ -st zig-zag does extend above  $b_{l-1}$ . Choose  $b_l$  to be the ball of  $\text{fw}(w)$  directly east of  $c_0$  (recall  $c_0$  is in the  $(i+1)$ -st zig zag). Now consider the semi-bounded reverse walk  $(b_l, \dots, b_{l+l'})$  with respect to  $C_k$  (the walk must terminate since there is no channel of  $\text{fw}(w)$  northwest of  $C_k$ ). By the same argument as in the end of the previous paragraph,  $(\text{fw}(d_w^{SW}))(b_{l+l'}) = d_{\text{fw}(w)}^{SW}(b_{l+l'}) = i + l'$ . The fact that the walk ended implies that the  $(i + l' + 1)$ -st zig-zag lies south of  $b_{l+l'}$  (as described in Example 14.4). Then  $T^{(i+l')}$  lies north of  $S^{(i+l'+1)}$ .  $\square$

Now we are ready to prove that  $\Phi(\widetilde{W}) \subseteq \Omega_{\text{dom}}$ .

*Proof of Theorem 5.10.* Of course it is sufficient to prove that dominance is preserved by every step of AMBC. Let  $w$  be a partial permutation. We can assume the Shi poset of  $w$  has at least two disjoint maximal antichains; otherwise dominance does not impose any restriction. Let  $S = \text{st}(w)$  and  $T = \text{st}(\text{fw}(w))$ ; we are trying to show that  $T$  is no lower than  $S$ . Consider two numberings of  $\text{fw}(w)$ :  $\text{fw}(d_w^{SW})$  and  $d_{\text{fw}(w)}^{SW}$  (chosen to coincide on the channels). The cells of  $S$  (resp.  $T$ ) have a natural numbering  $d_S : S \rightarrow \mathbb{Z}$  (resp.  $d_T : T \rightarrow \mathbb{Z}$ ) on them coming from  $d_w^{SW}$  (resp.  $d_{\text{fw}(w)}^{SW}$ ).

Recall that for any ball  $b$ , we have  $(\text{fw}(d_w^{SW}))(b) \geq d_{\text{fw}(w)}^{SW}(b)$ . This implies that the southwest (resp. northeast) ball of  $\text{fw}(w)$  labeled  $i$  by  $d_{\text{fw}(w)}^{SW}$  lies (weakly) east (resp. south) of the cell of  $S$  labeled  $\geq i$ . Hence for any  $i$ ,  $T^{(i)}$  lies southeast of  $S^{(i)}$ .

Let  $t$  be the partial permutation whose balls are the cells of  $T$  (as done in Section 13.2); we now show that the numbering  $d_T$  is precisely the backward numbering of  $t$  with respect to  $S$ . Since  $d_T$  is a monotone and  $T^{(i)}$  lies southeast of  $S^{(i)}$ , we know that  $d_T(b)$  is less than or equal to the backward numbering of  $b$  (recall Remark 13.1). By Lemma 14.12, there exists  $i$  such that  $b_0 = T^{(i)}$  lies north of  $S^{(i+1)}$ . Hence the backward numbering of  $b_0$  is at most  $i = d_T(b_0)$ ; hence the two numberings must coincide on  $b_0$ . However they both number the balls by consecutive integers, so if they coincide once then they coincide everywhere.

Now since  $b_0$  lies north of  $S^{(i+1)}$  the induced numbering of  $\text{bk}_S(t)$  is the southwest channel numbering. By Lemma 13.14,  $T$  is no lower than  $S$ , as desired.  $\square$

**14.4. Proof of bijectivity.** In the first part of this section we will develop the theory to show that for any  $w$ ,  $\Psi(\Phi(w)) = w$ .

**Theorem 14.13.** *Consider a partial permutation  $w$ , a channel  $C$ , and reverse zig-zags defined by the channel numbering  $d^C$ . Let  $S = \text{st}_C(w)$  and consider the partial permutation  $\text{fw}_C(w)$ . Then  $\text{fw}(d^C)$  coincides with the backward numbering of  $\mathcal{B}_{\text{fw}_C(w)}$  with respect to  $S$ .*

Assuming the above theorem it is clear that a step of the forward algorithm (with respect to any channel numbering) followed by a step of backward algorithm preserves the partial permutation.

*Proof of Proposition 5.2.* The sought result is an immediate corollary of Theorem 14.13.  $\square$

It follows that for any permutation  $w$ ,  $\Psi(\Phi(w)) = w$ . Now let us work toward the proof of Theorem 14.13.

**Definition 14.14.** Consider a partial permutation  $w$  and reverse zig-zags defined by a proper numbering  $d$ . A ball  $b \in \mathcal{B}_{\text{fw}_d(w)}$  on the zig-zag labeled  $i$  is *Nr-terminal* if the zig-zag labeled  $i + 1$  is completely south of  $b$ . Similarly,  $b$  is *Wr-terminal* if the zig-zag labeled  $i + 1$  is completely east of  $b$ .

A ball  $b \in \mathcal{B}_{\text{fw}_d(w)}$  on the zig-zag labeled  $i$  is *Nr-terminating* if there exists a reverse path  $(b = b_0, b_1, \dots, b_k)$  such that  $b_l$  is on the  $(i + l)$ -th zig-zag and  $b_k$  is *Nr-terminal* (so an *Nr-terminal* ball is always *Nr-terminating*). Define *Wr-terminating* analogously.

**Lemma 14.15.** *Consider a partial permutation  $w$  and zig-zags defined by a channel numbering  $d^C$ . Then every ball of  $\text{fw}_C(w)$  is either *Nr-terminating* or *Wr-terminating* (or both).*

*Proof.* Choose any zig-zag; let  $c$  be the southwestern end of it and  $c'$  be the northeastern end of it. Choose paths  $p = (c_0 = c, c_1, \dots, c_k)$  and  $p' = (c'_0 = c', c'_1, \dots, c'_k)$  with  $d^C(c_{j+1}) = d^C(c_j) - 1$ ,  $d^C(c'_{j+1}) = d^C(c'_j) - 1$ , and  $c_k = c'_k$  (that is, take paths of maximal worth from both endpoints to the channel and extend one of them along the channel until the paths meet). Let  $i = d^C(c_k)$ .

Choose a ball  $b$  of  $\text{fw}_C(w)$  in the  $i$ -th zig-zag. Suppose  $b$  is northeast of  $c_k$ . Consider a semi-bounded reverse path  $(b_0 = b, \dots, b_l)$  from  $b$  above  $p'$ ; it necessarily terminates since  $c'_0$  is the northeast-most ball of its zig-zag. As described in Example 14.4, this necessarily means that  $b_l$  is *Nr-terminal*. Now if  $b$  is southwest of  $c_k$  then the semi-bounded reverse walk leads to a *Wr-terminal* ball. Of course the paths  $p, p'$  could have been extended along  $C$  to an  $(n, n)$ -translate of any zig-zag, so the argument works for any ball  $b$ . This finishes the proof.  $\square$

*Proof of Theorem 14.13.* Notice that, by construction,  $\text{fw}(d^C)$  is a monotone numbering and that for each ball  $b \in \mathcal{B}_{\text{fw}_C(w)}$  with  $\text{fw}(d^C)(b) = i$ ,  $S_i$  lies northwest of  $b$ . By Remark 13.1, we know that for any ball  $b$ ,  $\text{fw}(d^C)(b)$  is at most equal to the backward numbering of  $b$ .

For an  $Nr$ - or  $Wr$ -terminal ball  $b$ , it is clear that the backward numbering is at most equal to  $\text{fw}(d^C)(b)$ , hence the two numberings must coincide on terminal balls. Now choose any  $b'$ ; by Lemma 14.15 it is terminating. Choose a reverse path from  $b'$  to a terminal ball such that  $\text{fw}(d^C)$  increases by 1 on each step. The backward numbering must increase by at least one at each step. Hence the backward numbering of  $b'$  is at most  $\text{fw}(d^C)(b')$ . This finishes the proof.  $\square$

We have now proven that for any  $w \in \widetilde{W}$ ,  $\Psi(\Phi(w)) = w$ . In the second part of the section we will show that if  $(P, Q, \rho) \in \Omega_{\text{dom}}$ , then  $\Phi(\Psi(P, Q, \rho)) = (P, Q, \rho)$ . Before getting to the proof we need to develop the theory of backward numberings slightly further.

The next major result shows that the induced numbering on a partial permutation resulting from a backward step is, in fact, a channel numbering. Before it we need a definition and a couple of lemmas.

**Definition 14.16.** Consider a partial permutation  $w$  and a compatible stream  $S$  (with some fixed proper numbering). Then  $b \in \mathcal{B}_w$  whose backward numbering is  $i$  is *N-terminal* if it lies north of  $S^{(i+1)}$ . Similarly, such  $b$  is *W-terminal* if it lies west of  $S^{(i+1)}$ .

A ball  $b$  is *N-terminating* if there exists a reverse path  $(b = b_0, \dots, b_k)$  such that the backward numbering increases by 1 at each step and  $b_k$  is *N-terminal*. Similarly for *W-terminating*.

Note that balls of  $w$  which are *N*- or *W*-terminal are precisely the ones whose numbering does not get bumped in the backward numbering algorithm.

**Lemma 14.17.** Consider a partial permutation  $w$  and a compatible stream  $S$  (with some fixed proper numbering). Then any  $b \in \mathcal{B}_w$  is either *N-terminating* or *W-terminating* (or both).

*Proof.* Suppose we start with a ball  $b_0$  of  $w$ . If it was not bumped then it is terminal and we are done. If it was bumped, let  $b_1$  be the ball responsible for the bumping (i.e. let  $b_1$  be the northwest ball southeast of  $b_0$  which had the same number as  $b_0$  just before  $b_0$  was bumped for the last time). If  $b_1$  is terminal, we are done. If not, repeat the process. Eventually the process must stop since the backward numbering algorithm has finitely many steps.  $\square$

We will see later (Lemma 16.14) that in case when some balls are both *N*- and *W*-terminating, all these balls lie in the same river.

**Lemma 14.18.** Consider a partial permutation  $w$  and a compatible stream  $S$  (with some fixed proper numbering). Let  $d$  be the induced proper numbering on  $\text{bk}_S(w)$  and consider the reverse zig-zags corresponding to it. Suppose two balls of  $w$ ,  $b$  and  $b'$ , are part of the same zig-zag,  $b$  is southwest of  $b'$ , and  $b$  is *N-terminating*. Then  $b'$  is also *N-terminating*.

*Proof.* Suppose  $p = (b_0 = b, \dots, b_k)$  is a reverse walk visiting consecutive zig-zags, and  $b_k$  is *N-terminal*. Consider the beginning of a semi-bounded (by  $p$ ) reverse walk starting at  $b'$ .

If the walk terminates in less than  $k$  steps, then its last ball is  $N$ -terminal as explained in Example 14.4. Otherwise we get a sequence of balls  $(b'_0, \dots, b'_k)$  and  $b'_k$  is north of  $b_k$  on the same zig-zag. Hence  $b'_k$  would also be  $N$ -terminal.  $\square$

Of course, in the notation of the lemma, any ball west of a  $W$ -terminating ball in the same zig-zag is also  $W$ -terminating.

**Proposition 14.19.** *Consider a partial permutation  $w$  and a compatible stream  $S$  (with some fixed proper numbering). Let  $d$  be the induced proper numbering on  $\text{bk}_S(w)$ . Then  $d$  is always a channel numbering.*

*Proof.* First note that if the flow of  $S$  is strictly greater than the width of the Shi poset of  $w$ , then  $\text{bk}_S(w)$  cannot have two disjoint channels (between two channels of  $\text{bk}_S(w)$  there is a channel of  $w$ ); so any proper numbering is necessarily a channel numbering. Hence assume the flow of  $S$  is equal to the width of the Shi poset of  $w$ .

We will find a ball  $c \in \mathcal{B}_{\text{bk}_S(w)}$  such that for any  $c'$  with  $d(c')$  sufficiently large, there exists a path  $(c_0 = c', \dots, c_k = c)$  with the value of  $d$  decreasing by 1 at each step. Let us first see that this will finish the proof. We will know that there exists such a path from some  $(n, n)$ -translate  $c'$  of  $c$ . By Lemma 11.4,  $c'$  is part of some channel  $C$ . Then  $d = d_{\text{bk}_S(w)}^C$ .

Choose  $c \in \mathcal{B}_{\text{bk}_S(w)}$  such that in its zig-zag, all the balls of  $w$  weakly north of it are  $N$ -terminating and all the balls of  $w$  weakly west of it are  $W$ -terminating. Suppose  $c$  is in the  $i$ -th zig-zag. Let  $b \in \mathcal{B}_w$  be the ball directly south of  $c$  and  $b' \in \mathcal{B}_w$  be the ball directly east of  $c$ . Choose a reverse path  $(b_0 = b, b_1, \dots, b_k)$  which visits consecutive zig-zags and has  $b_k$  being  $W$ -terminal. Similarly, choose a reverse path  $(b'_0 = b', b'_1, \dots, b'_l)$  which visits consecutive zig-zags and has  $b'_l$  being  $N$ -terminal.

Without loss of generality,  $l \geq k$ . First suppose  $l > k$ . Pick a ball  $c' \in \mathcal{B}_{\text{bk}_S(w)}$  in the  $(i + l + 1)$ -st zig-zag. Since  $b'_l$  is  $N$ -terminal, we know that  $b'_l$  is north of  $c'$ . If  $b'_l$  is west of  $c'$ , let  $c_1$  be the ball of  $\text{bk}_S(w)$  directly west of  $b'_l$ . Otherwise, let  $c_1$  be any ball northwest of  $c'$  with  $d(c_1) = i + l$  (it exists since  $d$  is proper). Thus  $c_1$  is northwest of  $c_0$ , strictly west of  $b'_l$ , and in the  $(i + l)$ -th zig-zag. Let  $(c_1, \dots, c_{l-k})$  be the beginning of a semi-bounded walk from  $c_1$  with respect to  $(b'_l, b'_{l-1}, \dots, b'_{k+1})$ . Let  $(c_{l-k}, c_{l-k+1}, \dots, c_{l+1})$  be the bounded walk from  $c_{l-k}$  between  $(b'_k, \dots, b'_0)$  and  $(b_k, \dots, b_0)$ . But between  $b_0$  and  $b'_0$  there is only one ball of  $\text{bk}_S(w)$ , namely  $c$ . So  $c_{l+1} = c$ . Of course from any zig-zag southwest of  $(i + l + 1)$ -st one we can find a path the case  $l > k$ . If  $l = k$  then the same argument works except in the construction of the path  $(c_0 = c', \dots, c_{l+1})$  one does not need to do the semi-bounded walk.  $\square$

In the next lemma we prove that under appropriate dominance assumptions the channel numbering in the previous lemma is actually the southwest channel numbering.

**Lemma 14.20.** *Consider a partial permutation  $w$  and a compatible stream  $S$  (with some fixed proper numbering). Let  $T = \mathbf{st}(w)$ . Suppose that  $S$  and  $T$  have the same flow and  $T$  is no lower than  $S$  (recall Definition 13.13). Let  $d$  be the numbering of  $\mathcal{B}_{\text{bk}_S(w)}$  induced by the backward step. Then  $d = d_{\text{bk}_S(w)}^{\text{SW}}$ .*

*Proof.* We know that  $d$  is a channel numbering; consider the balls of  $w$ , the balls of  $\text{bk}_S(w)$ , and the zig-zags defined by  $d$ . We need to show that for any ball  $c$  there is a path from  $c$  to the southwest channel of  $\text{bk}_S(w)$  such that  $d$  decreases by 1 at each step. Since  $d$  is proper, it is sufficient to show that such a path exists for any  $c$  in some fixed zig-zag.

Number  $T$  via its backward numbering with respect to  $S$ . Fixing a proper numbering of  $T$  is equivalent to fixing a particular shift of  $d_w^{SW}$ ; more precisely we fix a shift of  $d_w^{SW}$  so that the column of  $T^{(i)}$  is the column of the southwest ball  $b$  with  $d_w^{SW}(b) = i$ . Notice that  $d_w^{SW}$  is a monotone numbering which is bounded above by the stream numbering. Hence  $d_w^{SW}$  is a lower bound for the backward numbering of  $w$ .

Since  $T$  is no lower than  $S$ , there exists  $i$  for which  $T^{(i)}$  is north of  $S^{(i+1)}$ . Consider the ball  $b \in \mathcal{B}_w$  which is directly east of  $T^{(i)}$ . By definition,  $d_w^{SW}(b) = i$ . Since  $S^{(i+1)}$  is south of  $b$ , the backward numbering of  $b$  is at most  $i$ . Using the previous paragraph, we conclude that the backward numbering of  $b$  is  $i$ .

Construct a path  $(b_0 = b, b_1, \dots, b_k)$  to the southwest channel of  $w$  such that  $d_w^{SW}$  decreases by 1 at each step. The backward numbering must decrease at each step; since  $d_w^{SW}$  is a lower bound for it, the two must coincide on the whole path. Thus the path visited consecutive zig-zags. Continue this path by consecutive elements of the southwest channel:  $(b_{k+1}, b_{k+2}, \dots)$ .

Now consider  $c \in \mathcal{B}_{\text{bk}_S(w)}$  in the  $(i+1)$ -st zig-zag. There exists a ball  $c_1$  of  $\text{bk}_S(w)$  northwest of  $c$  in the  $i$ -th zig-zag and west of  $b$  (if  $c$  is east of  $b$  then it is the ball directly west of  $b$ ; otherwise it is any ball in the  $i$ -th zig-zag which is northwest of  $c$ ). Let  $(c_1, c_2, \dots)$  be a semi-bounded walk from  $c_1$  with respect to  $(b_0 = b, b_1, \dots)$ . This walk is necessarily infinite as explained in Example 14.4. Moreover, infinitely many cells of the path lie west of the corresponding entries of the southwest channel of  $w$ . Hence this path must intersect the southwest channel of  $\text{bk}_S(w)$ .  $\square$

Now it is easy to show that  $\Phi \circ \Psi$  is the identity map provided we start with a dominant weight.

*Proof of Theorem 5.11.* In each step of the backward algorithm we end up with a new partial permutation  $\text{bk}_S(w)$ . This partial permutation has two numberings of interest: the numbering induced by the backward step, and its own southwest channel numbering. If the flow of  $S$  is strictly greater than the width of the Shi poset of  $w$ , the two coincide since there is only one proper numbering up to shift. If the flow of  $S$  is equal to the width of the Shi poset of  $w$ , then the two coincide by the previous Lemma. So, the steps of AMBC will just be undoing the steps of the backward algorithm in reverse order.  $\square$

We have shown on page 50 that  $\Phi(w) \in \Omega_{\text{dom}}$ . Combined with the results of this section this proves that AMBC provides a bijection between  $\widetilde{W}$  and  $\Omega_{\text{dom}}$ .

## 15. DISTANCES AND ALTITUDES

The main point of this section is to interpret distances between channels of  $w$  in terms of weights and in terms of distances between channels of  $\text{fw}(w)$ . Namely, we will prove Theorem 8.1.

Let  $w$  be a partial permutation whose Shi poset has  $k \geq 2$  disjoint maximal antichains. Consider interlacing maximal collections of channels  $\{C_1, \dots, C_k\}$  and  $\{D_1, \dots, D_{k-1}\}$  (as described in Corollary 14.6).

**Theorem 15.1.** *If  $k \geq 3$ , then for any  $1 \leq i \leq k-2$ ,  $h(D_i, D_{i+1}) = h(C_{i+1}, C_{i+2})$ .*

*Proof.* Fix a shift of  $d_w^{SW}$ . Throughout the proof we will be considering the reverse zig-zags determined it. Fix  $1 \leq i \leq k-2$ . All further numberings of  $\mathcal{B}_w$  will be chosen to coincide with

$d_w^{SW}$  on  $C_{i+2}$  and all further numberings of  $\mathcal{B}_{fw(w)}$  will be chosen to coincide with  $fw(d_w^{SW})$  on  $D_{i+1}$ . This fixes the numbering of  $C_{i+2}$  and  $D_{i+1}$  for the rest of the proof.

By Proposition 14.11 we know that  $fw(d_w^{SW})$  and  $d_{fw(w)}^{SW}$  coincide on all channels of  $fw(w)$ . Since  $D_i$  is southwest of  $D_{i+1}$ ,  $d_{fw(w)}^{SW}$  and  $d_{fw(w)}^{D_i}$  coincide on  $D_{i+1}$ . Hence  $fw(d_w^{SW})$  and  $d_{fw(w)}^{D_i}$  coincide on  $D_{i+1}$ . Suppose for some  $c \in C_{i+1}$ , we have  $d_w^{C_{i+2}}(c) = l$ . We will show that in this case, for the element  $b \in D_i$  in the reverse zig-zag of  $c$  we have  $d_{fw(w)}^{D_{i+1}}(b) = l$ . This will finish the proof.

First we will prove that  $d_{fw(w)}^{D_{i+1}}(b) \geq l$ . By assumption, there is a path  $p = (c_0 = c, \dots, c_h)$  from  $c$  to  $C_{i+2}$  of worth  $l$  (so the fixed numbering of  $c_h$  must be  $l - h$ ). We will build a reverse path from  $D_{i+1}$  to  $D_i$  (see left side of Figure 28). Start with a ball  $b_0 \in D_{i+1}$  which is on the same reverse zig-zag as  $c_h$ . Let  $(b_0, b_1, \dots, b_{h-1})$  be the funnel reverse walk between  $p$  and  $C_{i+1}$ . Let  $b_h \in \mathcal{B}_{fw(w)}$  be the ball of  $fw(w)$  directly south of  $c$ . Let  $(b_h, b_{h+1}, \dots, b_{h+h'})$  be a reverse bounded walk between  $C_i$  and  $C_{i+1}$  to  $D_i$ . Thus we have a path with  $h + h'$  steps from  $b_{h+h'} \in D_i$  to  $b_0 \in D_{i+1}$ . The fixed numbering of  $b_0$  is  $l - h$  (it is the same as the numbering of  $c_h$ ), so the worth of the path is  $l + h'$ . Thus  $d_{fw(w)}^{D_{i+1}}(b_{h+h'}) \geq l + h'$ . Now  $b_{h+h'}$  is  $h'$  reverse zig-zags ahead of  $b$ . So  $d_{fw(w)}^{D_{i+1}}(b) \geq l$ .

Now we will prove that  $d_{fw(w)}^{D_{i+1}}(b) \leq l$ ; the argument is nearly symmetric. Suppose there is a path  $q = (b_0 = b, \dots, b_g)$  from  $b$  to  $D_{i+1}$  of worth  $s$  (so the fixed numbering of  $b_g$  must be  $s - g$ ). We will build a path from  $c$  to  $C_{i+2}$  (see right side of Figure 28). Let  $(c_0 = c, \dots, c_{g-1})$  be the funnel walk between  $D_{i+1}$  and  $q$ . Let  $c_g \in \mathcal{B}_w$  be the ball of  $w$  directly north of  $b_g$ . Let  $(c_g, \dots, c_{g+g'})$  be the walk from  $c_g$  to  $C_{i+2}$  (if  $D_{i+2}$  exists, we can find a bounded walk between  $D_{i+1}$  and  $D_{i+2}$ ; otherwise we can use a semi-bounded walk). The fixed numbering of  $c_{g+g'}$  is  $s - g - g'$  and the worth of the path  $(c_0, \dots, c_{g+g'})$  is  $s$ . Hence  $s \leq d_w^{C_{i+2}}(c) = l$ . So  $d_{fw(w)}^{D_{i+1}}(b)$ , which is the maximal value of  $s$  over the set of paths, is also no larger than  $l$ .  $\square$

Before proving the statement for the southwest two channels  $C_1, C_2$ , we give an alternative description of the pseudometric  $h$  on the set of channels:

**Lemma 15.2.** *Suppose  $w$  is a partial permutation and  $C, C'$  are channels.*

$$h(C, C') = \min_{(b_0, \dots, b_f)} d^C(b_0) - (d^{C'}(b_f) + f)$$

where the minimum is taken over all paths from  $C$  to  $C'$ .

*Proof.* Note that if we fix  $b_0$  then the right hand side becomes  $d^C(b_0) - d^{C'}(b_0)$ , where the shift of  $d^{C'}$  is chosen so that  $d^C$  and  $d^{C'}$  coincide on  $C'$ . This is independent of  $b_0$  and is, by definition,  $h(C, C')$ .  $\square$

**Theorem 15.3.** *Let  $S$  be the stream resulting from a step of AMBC on  $w$ , and let  $T$  be the stream resulting from a step of AMBC on  $fw(w)$ . Choose  $a_0$  to be the altitude of the stream from the class of  $T$  (recall Definition 3.22) which is concurrent to  $S$ . Then  $a(T) = a_0 + h(C_1, C_2)$*

*Proof.* We have already proved (Theorem 5.10) that  $a(T) \geq a_0$ . Suppose for some  $l$ ,  $a(T) = a_0 + l$ . We will show that  $h(C_1, C_2) \leq l$  and that  $h(C_1, C_2) \geq l$ . To prove the first statement we need to find a path  $(b_0, \dots, b_f)$  from  $C_1$  to  $C_2$  such that  $d^{SW}(b_0) - (d^C(b_f) + f) \leq l$ .

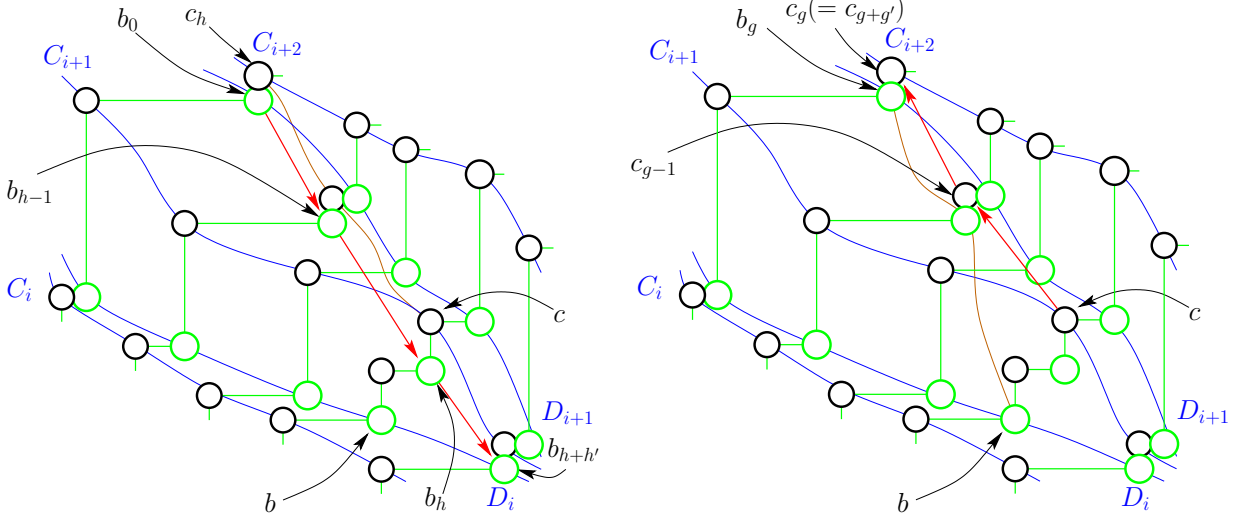


FIGURE 28. The two paths constructed in the proof of Theorem 15.1. In the right half the ball  $c_g$  already lies on the channel  $C_{i+2}$ , so the last walk is not necessary in the example.

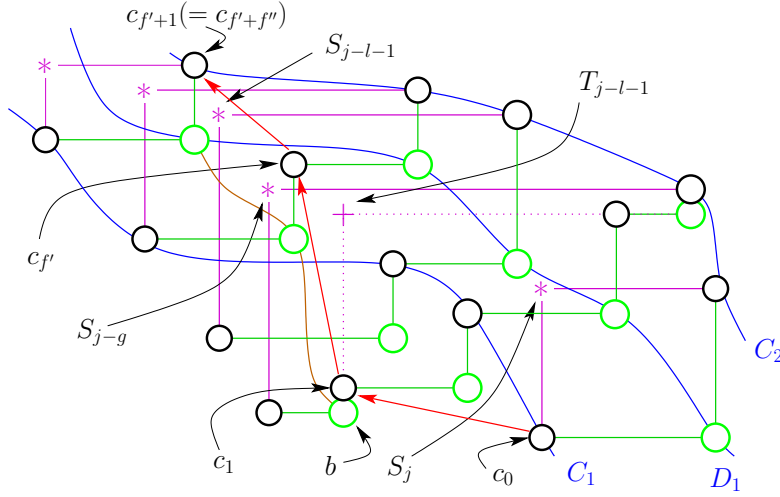


FIGURE 29. The construction of the first path in the proof of Theorem 15.3. The cells of  $S$  are labeled by  $*$ 's while the cell of  $T$  is labeled by  $+$  (other cells of  $T$  are not relevant hence are omitted).

Fix some proper numbering of  $S$ . If  $t$  is the partial permutation whose balls are the cells of  $T$ , fix the numbering  $T$  which gives the backward numbering of  $t$  with respect to  $S$ . Recall from the proof of Theorem 5.10 on page 50 that choosing the relative matching of the numberings of  $T$  and  $S$  in this way ensures that  $\text{fw}(d_w^{SW})$  and  $d_{\text{fw}(w)}^{SW}$  coincide on channels.

One can see that  $a(T) = a_0 + l$  means precisely that for some  $j$ ,  $S_j$  is east of  $T_{j-l-1}$ . Now  $T_{j-l-1}$  lies directly north of a ball  $b \in \mathcal{B}_{\text{fw}(w)}$  (see Figure 29). So  $d_{\text{fw}(w)}^{SW}(b) = j - l - 1$  and



hence  $(\text{fw}(d_w^{SW})) (b) \geq j - l - 1$  (Proposition 14.11). Let  $g = j - (\text{fw}(d_w^{SW})) (b)$ ; this way  $b$  is in the  $(j - g)$ -th zig-zag. We know that  $0 < g \leq l + 1$ .

Consider a path  $p = (b_0 = b, \dots, b_{f'})$  of maximal worth from  $b$  to  $D_1$ . We will now construct the required path from  $C_1$  to  $C_2$ . Choose  $c_0 \in C_1$  such that  $d_w^{SW}(c_0) = j$ . Let  $c_1 \in \mathcal{B}_w$  be a ball northwest of  $c_0$  and between  $b$  and  $D_1$  (it exists by Lemma 14.1). Let  $(c_1, \dots, c_{f'})$  be the funnel walk between  $p$  and  $D_1$ . Let  $c_{f'+1} \in \mathcal{B}_w$  be the ball directly north of  $b_{f'}$ . Let  $(c_{f'+1}, \dots, c_{f'+f''})$  be the walk from  $c_{f'+1}$  to  $C_2$  (it is either bounded between  $D_1$  and  $D_2$  or semi-bounded above  $D_1$ , depending on whether  $k > 2$  or  $k = 2$ ).

The resulting path  $(c_0, \dots, c_{f'+f''})$  has  $f' + f''$  steps. By construction  $c_0$  had label  $j$  (in the SW numbering) and  $c_1$  had label  $j - g$ . Now the label of  $b_{f'}$  must be  $j - g - f'$ , and so the label of  $c_{f'+1}$  is also  $j - g - f'$ . Finally the label of  $c_{f'+f''}$  must be  $j - g - f' - f'' + 1 \geq j - l - f' - f''$ . The worth of the path  $(c_0, \dots, c_{f'+f''})$  with respect to  $C_2$  is thus  $\geq j - l$ . Thus  $h(C_1, C_2) \leq l$  as desired.

Now we will show that  $h(C_1, C_2) \geq l$ . Suppose  $h(C_1, C_2) = l'$ . Choose a path  $q = (c_0, c_2, \dots, c_f)$  from  $C_1$  to  $C_2$  such that  $d_w^{SW}(c_0) - (d_w^{SW}(c_f) + f) = l'$ . We want to show that for some  $j$ ,  $S^{(j)}$  is east of  $T^{(j-l'-1)}$ . Let  $b_0$  be the ball of  $D_1$  on the zig-zag of  $c_f$ ; we will construct a reverse path  $(b_0 = b, \dots, b_{f'})$ . First consider a funnel walk  $(b_0, \dots, b_{f-1})$  between  $q$  and  $C_1$ . If  $c_0$  is the SW-most ball of its zig-zag, then let  $f' = f - 1$ . Otherwise let  $b_f$  be the ball of  $\text{fw}(w)$  directly south of  $c_0$ . Then continue the reverse path to a (necessarily finite) semi-bounded walk  $(b_f, b_{f+1}, \dots, b_{f'})$  with respect to  $C_1$ .

Suppose the zig-zag containing  $b_{f'}$  is labeled  $j - 1$ . Notice that  $j - 1 = d_w^{SW}(c_0) + f'$  since the walk visits each zig-zag. Since the walk stopped, we must have  $S_j$  lying east of  $b_{f'}$ . We know that  $d_{\text{fw}(w)}^{SW}(b_0) = d_w^{SW}(c_f)$  since  $b_0$  is on a channel. Hence  $d_{\text{fw}(w)}^{SW}(b_{f-1}) \geq d_w^{SW}(c_f) + f - 1 = d_w^{SW}(c_0) - l' - 1$ . Hence  $d_{\text{fw}(w)}^{SW}(b_{f-1}) \geq d_w^{SW}(c_0) - l' + f' = j - l' - 1$ . Thus  $T^{(j-l'-1)}$  is west of  $b_{f'}$ . This finishes the proof.  $\square$

## 16. PROOF OF WEYL SYMMETRY

The purpose of this section is to prove Theorem 6.3 and Corollary 6.4, namely show that  $\Phi \circ \Psi$  preserves the tabloids and maps the weight to the dominant chamber.

Suppose  $P$  and  $Q$  are two tabloids with the same shape  $\lambda$ , and  $\rho$  is an integer vector of size  $\ell(\lambda)$ . Let  $r_1 = 0$ . For each  $2 \leq i \leq \ell(\lambda)$ : if  $\lambda_i < \lambda_{i-1}$  then let  $r_i = 0$ , otherwise let  $r_i$  be the unique integer such that  $\text{st}_{r_i}(P_i, Q_i)$  is concurrent to  $\text{st}_0(P_{i-1}, Q_{i-1})$ .

Choose  $i$  such that  $\lambda_i = \lambda_{i-1}$ . Since any permutation is a product of transpositions, to prove Theorem 6.3 it suffices to prove that  $\Psi(P, Q, \rho) = \Psi(P, Q, \rho')$ , where

$$\rho' = (\rho_1, \dots, \rho_{i-2}, \rho_i - r_i + r_{i-1}, \rho_{i-1} - r_{i-1} + r_i, \rho_{i+1}, \dots, \rho_{\ell(\lambda)}).$$

We will in fact prove that after the two backward steps which remove rows  $i$  and  $i - 1$ , the partial permutations already coincide.

The notions of shifting streams, channels, and rivers will be central to the rest of this section. Recall the for a stream  $S$  we write  $S \langle i \rangle$  for the shift of  $S$  by  $i$ , and analogously for partial permutations consisting of a single channel.

**Definition 16.1.** For a partial permutation  $w$ , a channel  $C$ , and a river  $R$  we write

- $w \langle i \rangle_C$  for the permutation obtained from  $w$  by replacing  $C$  with  $C \langle i \rangle$ ,

- $w \langle i \rangle_R$  for the permutation obtained from  $w$  by either shifting the northeast channel of  $R$  by  $i$  if  $i > 0$  or shifting the southwest channel of  $R$  by  $i$  if  $i < 0$ .

In Section 16.1 we study what happens to the backward step of a permutation if we change the altitude of the stream by 1. In the Section 16.2 we study what happens when change the altitude of one stream and do two steps back (preserving the altitude of the second stream). In Section 16.3 we combine the two theories to say that first decreasing the altitude of one stream appropriately and then increasing the altitude of the other preserves the partial permutation resulting from the two backward steps.

**16.1. Shifting streams.** Consider a partial permutation  $w$  and compatible streams  $S$  and  $S' = S \langle 1 \rangle$ . Assume that the streams have the same flow as the width of the Shi poset of  $w$ . In this section we describe the difference between the partial permutations  $\text{bk}_S(w)$  and  $\text{bk}_{S'}(w)$ ; it turns out that only one channel is different between them.

We will have to consider proper numberings of the two streams  $S$  and  $S'$ . The overall shift is unimportant, so we fix some proper numbering of  $S$ . There are two natural numberings of  $S'$ : one matches that of  $S$  on the rows (call it the *row-matching numbering*) while the other one on the columns (call it the *column-matching numbering*).

**Lemma 16.2.** *Let  $d$  be the backward numbering of  $w$  with respect to  $S$ , and  $d'$  the backward numbering with respect to the row-matching numbering of  $S'$ . Then for any  $b \in \mathcal{B}_w$  we have  $d(b) = d'(b)$  or  $d(b) = d'(b) + 1$ .*

*Proof.* Let  $d''$  be the backward numbering with respect to the column-matching numbering of  $S'$ . It is clear that  $d'(b) \leq d(b)$  (since  $d'$  is a monotone numbering which is no larger than the stream numbering for  $S$ ). For the same reason, it is clear that  $d(b) \leq d''(b)$ . Now by construction  $d''(b) - d'(b) = 1$ . This finishes the proof.  $\square$

Recall the notions of  $N$ -terminating and  $W$ -terminating balls of  $w$  from Definition 14.16. Using these notions we can be more precise about the balls on which the numberings disagree.

**Lemma 16.3.** *Let  $d$  be the backward numbering of  $w$  with respect to  $S$ , and  $d'$  the backward numbering with respect to the row-matching numbering of  $S'$ . Then  $d(b) = d'(b) + 1$  if and only if  $b$  is  $W$ -terminating with respect to  $S$ . Moreover, any ball satisfying these conditions is  $W$ -terminating with respect to  $S'$ .*

*Proof.* Suppose  $b$  is  $W$ -terminating with respect to  $S$ . Consider a reverse path  $(b_0 = b, b_1, \dots, b_k)$  such that  $d$  increases by 1 at each step and  $b_k$  is  $W$ -terminal. By construction,  $d'(b_k) < d(b_k)$ . So  $d'(b) < d(b)$ . Hence  $d'(b) = d(b) - 1$ , as desired.

Suppose  $d'(b) < d(b)$ . Consider a reverse path  $(b_0 = b, \dots, b_k)$  such that  $d'$  increases by one at each step and  $b_k$  is terminal with respect to  $S'$ . Of course,  $d$  must also increase at each step, hence  $d'(b_k) < d(b_k)$ . By the previous lemma,  $d'(b_k) = d(b_k) - 1$  and  $d$  increases by 1 at each step. To finish the proof we will show that  $b_k$  is  $W$ -terminal with respect to  $S$ .

Let  $i = d(b_k)$ ; we know that  $S^{(i)}$  is northwest of  $b_k$ . However since  $b_k$  was terminal with respect to  $S'$  and  $d'(b_k) = i - 1$ ,  $S'^{(i)}$  is not northwest of  $b_k$ . Since  $S'^{(i)}$  is in the same row as  $S^{(i)}$ , it is north of  $b_k$ . Hence  $S'_i$  is east of  $b_k$  (hence  $b_k$  is  $W$ -terminal with respect to  $S'$ ). Hence  $S_{i+1}$  is east of  $b_k$ . Thus  $b_k$  is  $W$ -terminal with respect to  $S$  as desired.  $\square$

**Lemma 16.4.** *Consider the permutations  $w$  and  $\text{bk}_S(w)$ , and the corresponding zig-zags. Recall that the balls of  $w$  in each zig-zag southwest of a certain point are  $W$ -terminating and*

northeast of that point are not  $W$ -terminating. For each  $i$ , let  $c_i$  be the ball of  $\text{bk}_S(w)$  in the  $i$ -th zig-zag directly north of the northeastern  $W$ -terminating ball of the zig-zag (or, if the zig-zag has no  $W$ -terminating balls, the southwest ball of the zig-zag). Then  $\{c_i\}_{i \in \mathbb{Z}}$  is a channel of  $\text{bk}_S(w)$ .

*Proof.* It is sufficient show that  $c_{i+1}$  is always southeast of  $c_i$ . Suppose first that  $c_{i+1}$  is north of  $c_i$ . Since the induced numbering of  $\text{bk}_S(w)$  is proper, there must be a ball in the  $i$ -th zig-zag which is strictly north of  $c_i$ . In particular, there exists a ball  $b$  of  $w$  directly east of  $c_i$ . Now  $c_{i+1}$  must be east of  $b$  (again, otherwise the induced numbering would not be proper). By construction,  $b$  was not  $W$ -terminating and in particular not  $W$ -terminal. Hence the southwest ball of  $\text{bk}_S(w)$  in the  $(i+1)$ -st zig-zag must be west of  $b$ . It cannot lie northwest of  $b$ , so it must be south of  $b$ . By the reflection of Lemma 14.1 (the “Key Lemma”) in an anti-diagonal, we know that there is a ball  $b'$  of  $w$  in the  $(i+1)$ -st zig-zag, southwest of  $c_{i+1}$ , and southeast of  $b$ . Now  $b'$  must be  $W$ -terminating since it lies below  $c_{i+1}$  in its zig-zag. Hence  $b$  is also  $W$ -terminating. This is a contradiction. Thus  $c_{i+1}$  is south of  $c_i$ .

Now if  $c_{i+1}$  were west of  $c_i$  then we could transpose the whole picture in the main diagonal and get a situation with  $c_{i+1}$  north of  $c_i$ . Thus this case follows by symmetry.  $\square$

**Definition 16.5.** Suppose  $w$  is a partial permutation and  $S$  is a compatible stream. Then a step of the backward algorithm induces a channel numbering on  $\text{bk}_S(w)$ . Call the river consisting of such channels the *indexing river* corresponding to  $S$  and  $w$ .

*Remark 16.6.* Recall the proof of Proposition 14.19 which stated that the induced backward numbering is a channel numbering. In it we concluded that any ball of  $\text{bk}_S(w)$  such that directly south of it is a  $W$  terminating ball and directly east of it is an  $N$ -terminating ball must lie on the indexing river.

**Lemma 16.7.** Consider the permutations  $w$  and  $\text{bk}_S(w)$ , and the corresponding zig-zags. Then the channel  $C = \{c_i\}_{i \in \mathbb{Z}}$  described in the previous lemma is the northeast channel of the indexing river.

*Proof.* By construction of  $C$  and the previous remark, it is clear that  $C$  is part of the indexing river. Consider a channel  $C'$  weakly northeast of  $C$  and ball  $c \in C'$  strictly northeast of some  $c_i$  and in the  $i$ -th zig-zag. We would like to show that  $d(C, C') > 0$ .

Suppose that this was not the case. Choose a path  $p = (c'_0, c'_1, \dots, c'_k = c)$  which visits consecutive zig-zag and starts at  $c'_0 \in C$ . Consider a ball  $b$  of  $w$  which is directly south of  $c$ . Since  $c$  is strictly northeast of  $c_i$ ,  $b$  is not  $W$ -terminating. Consider a reverse funnel walk  $(b_0 = b, b_1, \dots, b_{k-1})$  between  $C$  and  $p$ . We will show that  $b_{k-1}$  is  $W$ -terminating, thus arriving at the desired contradiction.

Either  $c'_0$  is a southwest element of its zig-zag, or there exists a ball  $b_k$  of  $w$  directly south of it. In the first case,  $b_{k-1}$  is, in fact  $W$ -terminal. In the second, let  $b_k$  be the ball directly south of  $c'_0$ . Then  $b_k$  is (by construction)  $W$ -terminating and is southeast of  $b_{k-1}$ ; thus  $b_{k-1}$  is also  $W$ -terminating. This finishes the proof.  $\square$

It is easy to see that the change in backward numbering described earlier results in the shift of  $C$  by 1:

**Theorem 16.8.** Consider a partial permutation  $w$  and compatible streams  $S$  and  $S' = S \langle 1 \rangle$ . Assume that the streams have the same flow as the width of the Shi poset of  $w$ . Let  $R$  (resp.

$R'$ ) be the indexing river of  $\text{bk}_S(w)$  (resp.  $\text{bk}_{S'}(w)$ ). Then  $\text{bk}_{S'}(w) = \text{bk}_S(w) \langle 1 \rangle_R$ . By transposing with respect to the main diagonal, we see that  $\text{bk}_S(w) = \text{bk}_{S'}(w) \langle -1 \rangle_{R'}$ .

*Remark 16.9.* As noted in Lemma 16.3, a ball which is  $W$ -terminating with respect to  $S$  remains  $W$ -terminating with respect to  $S'$ . It is also clear that any ball which is not  $W$ -terminating with respect to  $S$  remains  $N$ -terminating with respect to  $S'$  (the same path to an  $N$ -terminal ball works in both cases). Thus, as was recently mentioned in Remark 16.6 the new channel of  $\text{bk}_{S'}(w)$  is part of the indexing river of  $\text{bk}_{S'}(w)$ . Thus if we were to increase the altitude of the stream again the result of the backward step would be to shift by 1 the river containing the new channel.

**16.2. Shifting channels.** Consider a permutation  $w$  and a compatible stream  $S$  such that the flow of  $S$  is equal to the width of the Shi poset of  $w$ . Choose a channel  $C$  of  $w$  which is the northeast channel of its corresponding river. Let  $w' = w \langle 1 \rangle_C$ . In this section we are interested in studying the difference between the partial permutations  $\text{bk}_S(w)$  and  $\text{bk}_S(w')$ . It turns out that, just as in the case of shifting a stream, these permutations are related by shifting a channel.

**16.2.1. Preliminary results on shifting rivers.** We start with a few results about the relationship between  $w$  and  $w'$  themselves.

**Lemma 16.10.** *The Shi posets of  $w$  and  $w'$  have the same width.*

*Proof.* First note that the two areas shaded in Figure 30 cannot contain any balls of  $w$  or  $w'$ . For the lighter area it is true because  $C$  was a channel. For the darker area it is true since the existence of such a ball would contradict the fact that  $C$  is the northeast channel of its river (just replace one ball of  $C$  with a ball from the corresponding dark region).

It is sufficient to construct a proper numbering  $d$  of  $\mathcal{B}_{w'}$  which has the same period as  $d_w^C$ . Let  $C' = C \langle 1 \rangle$  (these are dashed in the figure). Any ball in  $\mathcal{B}_{w'}$  is either in  $C'$ , strictly southwest of  $C'$  (and, in fact, of  $C$ ), or strictly northeast of  $C'$ . For each  $b'$  in  $C'$ , let  $b$  denote the ball of  $C$  directly west of it. Define for each  $b' \in \mathcal{B}_{w'}$ ,

$$d(b') = \begin{cases} d_w^C(b') & \text{if } b' \text{ is strictly northeast of } C' \\ d_w^C(b) & \text{if } b' \in C' \\ d_w^C(b) - 1 & \text{if } b' \text{ is strictly southwest of } C' \end{cases}$$

One easily checks most of the conditions that ensure that  $d$  is proper; we will only do the more difficult cases. Continuity does not present problems, but with regard to monotonicity two cases deserve attention. First, suppose  $b'_1$  is northeast of  $C'$ ,  $b'_2 \in C'$ , and  $b'_1$  is northwest of  $b'_2$ . We can check that  $d(b'_1) < d(b'_2)$ . Suppose not. Consider a minimal worth path  $(c_0 = b'_1, c_1, \dots, c_k)$  from  $b'_1$  to  $C$  in  $w$ . If  $c$  is the ball directly south of  $b'_2$  then  $(c, c_0, \dots, c_k)$  is a path in  $w$  from  $C$  to itself which visits every zig-zag (with respect to  $d_w^C$ ). Moreover this path intersects  $C$  and contains balls strictly northeast of  $C$ . This contradicts the assumption that  $C$  was the northeast channel of its river. The second case is similar except  $b'_2$  is southwest of  $C'$ . In this case consider the ball  $b''_2$  of  $C'$  directly east of the ball of  $C$  on the zig-zag of  $b'_2$ . Replacing  $b'_2$  with  $b''_2$  reduces the proof to the previous case.  $\square$

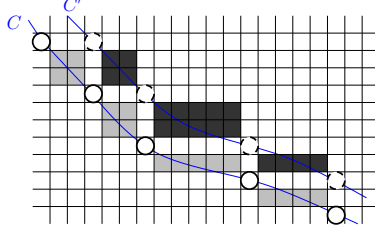


FIGURE 30. Two areas that cannot have any balls of  $w$  or  $w'$ .

*Remark 16.11.* The numbering  $d$  constructed in the proof of the last lemma is easily seen to be the channel numbering of  $w'$  with respect to  $C'$  (it is a proper numbering and from any ball there is a path to  $C'$  with  $d$  decreasing by 1 at each step).

**Lemma 16.12.** *Let  $C' = C \langle 1 \rangle$ . Then  $C'$  is the southwest channel of its river.*

*Proof.* Consider the zig-zags corresponding to  $d_{w'}^{C'}$ . From the previous lemma and remark we know the relationship between  $d_{w'}^{C'}$  and  $d_w^C$ . Suppose there exists a ball  $c$  of  $w'$  southwest of  $C'$  and part of the same river. Then there should be a path  $(c_0, c_1, \dots, c_k = c)$  from  $C'$  to  $c$  which visits consecutive zig-zags. We may assume that  $c_0 \in C'$  and  $c_1$  is southwest of  $C'$ . Let  $b$  be the element of  $C$  directly west of  $c_0$ . In this case,  $c_1$  is northwest of  $b$ . However,

$$d_w^C(b) = d_{w'}^{C'}(c_0) = 1 + d_{w'}^{C'}(c_1) = d_w^C(c_1).$$

This contradicts the assumption that  $d_w^C$  is proper.  $\square$

The next series of results further develops the theory of  $W$ - and  $N$ -terminating balls.

**Lemma 16.13.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . For any river  $R$  of  $w$  and any ball  $b \in R$ , if  $b$  is  $W$ -terminating (resp.  $N$ -terminating) then any ball of  $R$  is also  $W$ -terminating (resp.  $N$ -terminating).*

*Proof.* Let  $d$  be the backward numbering of  $w$  with respect to  $S$ . Consider a channel  $C$  with  $b \in C$  and another channel  $C' \subseteq R$ . By definition of a river, there exists a path  $p$  from  $b$  to an  $(n, n)$ -translate of  $b$  which intersects  $C'$  and at each step  $d_w^R$  decreases by one. Since at each step the backward numbering has to decrease and  $d$  has the same period as  $d_w^R$ , we know that  $d$  also decreased by one at each step of  $p$ . This means there is a reverse path from  $C'$  to  $b$  such that  $d$  decreases by one at each step. This finishes the proof.  $\square$

**Lemma 16.14.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . Then there exists at most one river whose balls are both  $W$ -terminating and  $N$ -terminating.*

This lemma follows easily from:

**Lemma 16.15.** *Consider a partial permutation  $w$  and a compatible stream  $S$ . Suppose  $B$  is a channel whose balls are both  $W$ -terminating and  $N$ -terminating. Fix the shift of  $d_w^B$  to coincide with the backward numbering on  $B$ . Then  $d_w^B$  and the backward numbering coincide on any channel of  $w$ .*

*Proof.* In this problem the zig-zags considered will be associated to the induced numbering on  $\text{bk}_S(w)$ . Consider another channel  $B'$ . Pick  $b \in B$  and  $b' \in B'$  on the same zig-zag. Without loss of generality,  $b'$  is northeast of  $b$ . Consider a reverse path  $(b_0 = b, b_1, \dots, b_k)$

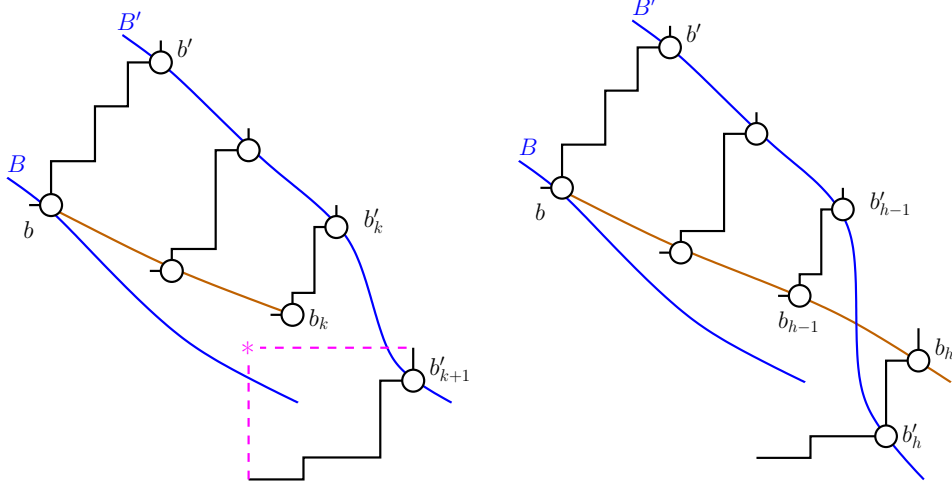


FIGURE 31. The first two cases considered in Lemma 16.15.

which visits consecutive zig-zags from  $b$  to a  $N$ -terminal ball  $b_k$ . Consider a reverse path  $(b'_0 = b', \dots, b'_{k+1})$  consisting of consecutive elements of the channel  $B'$ .

There are three possible cases:

- (1)  $b'_i$  is northeast of  $b_i$  for any  $i$  (see the left hand side of Figure 31),
- (2) the two paths cross without intersecting (see the right hand side of Figure 31),
- (3) the two paths intersect.

In the first case,  $b'_{k+1}$  is southeast of  $b_k$  (since it is east of  $b'_k$  and  $b_k$  is  $N$ -terminal). Then the path  $(b'_{k+1}, b_k, \dots, b_0)$  is a path from  $B'$  to  $B$  such that the backward numbering decreases by 1 at each step. Thus the backward numbering agrees with  $d_w^B$  on  $B'$ . In the last two cases (see the right hand side of Figure 31), let  $h$  be the smallest number such that  $b'_h$  is weakly southwest of  $b_h$ . Then  $(b'_h, b_{h-1}, \dots, b_0)$  is a path from  $B'$  to  $B$  such that the backward numbering decreases by 1 at each step. Thus the backward numbering again agrees with  $d_w^B$  on  $B'$ . □

We will periodically need to keep careful track of how a (reverse) path can “cross” a channel. To this extent we describe several types of interaction.

**Definition 16.16.** Suppose  $p = (c_0, c_1, \dots, c_k)$  is a path or a reverse path, and  $C$  is a channel. Then

- $p$  *bridges*  $C$  on step  $i$  if  $c_i$  is on one side of  $C$  (northeast or southwest) while  $c_{i+1}$  is on the other,
- $p$  *fords*  $C$  if for some  $1 \leq i < j < k$  we have  $c_{i+1}, \dots, c_j \in C$ , while  $c_i$  and  $c_{j+1}$  are on different sides of  $C$ .
- $p$  *skims*  $C$  if for some  $1 \leq i < j < k$  we have  $c_{i+1}, \dots, c_j \in C$ , while  $c_i$  and  $c_{j+1}$  are on the same side of  $C$ .
- $p$  *intersects*  $C$  if  $p$  fords or skims  $C$ .

We will need a non-algorithmic alternative definition for the backward numbering.

**Lemma 16.17.** Consider a partial permutation  $w$  and a compatible stream  $S$ . Let  $d$  be the backward numbering and  $d_0$  be the stream numbering. Then for any ball  $b$  of  $w$ ,

$$d(b) = \min_{(b_0=b, \dots, b_k)} d_0(b_k) - k,$$

where the minimum is taken over all reverse paths starting at  $b$ .

*Proof.* Let  $d'$  be the backward numbering in question. Suppose  $(b_0 = b, \dots, b_k)$  is a reverse path achieving the minimum in question. Then  $d'$  must increase at each step and  $d'(b_k) \leq d_0(b_k)$ . So  $d'(b) \leq d(b)$ . Now let  $(b_0 = b, \dots, b_k)$  be a reverse path to a terminal ball. Then  $d'(b) = d_0(b_k) - k$ . Thus by definition  $d(b) \leq d'(b)$ .  $\square$

**Definition 16.18.** For a reverse path  $(b_0 = b, \dots, b_k)$  as in the above lemma, we refer to the number  $d_0(b_k) - k$  as the  $r$ -worth of the path.

We now analyze the backward numbering with respect to  $S$  of  $w$  and of  $w'$  (the resulting strange numbering seen below miraculously results in shifting the channel  $\tilde{C}$  by 1 after the backward step, as will be seen in Figure 33).

**Theorem 16.19.** Let  $w$  be a partial permutation and  $S$  be a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Choose a non- $N$ -terminating river of  $w$  and let  $C$  be its northeast channel. Let  $w' = w \langle 1 \rangle_C$ . Let  $C' = C \langle 1 \rangle$ . Let  $\tilde{C}$  be the northeast channel of  $\text{bks}_S(w)$  which is southwest of  $C$ . Let  $d$  (resp.  $d'$ ) be the backward numbering of  $w$  (resp.  $w'$ ) with respect to  $S$ . Then

- for any  $b \in C$ ,  $d(b) = d'(b')$ , where  $b'$  is the ball of  $C'$  directly north of  $b$ ,
- for any  $b \in \mathcal{B}_w$  which is strictly northeast of  $C$  (and hence  $C'$ ),  $d'(b) = d(b)$ ,
- for any  $b \in \mathcal{B}_w$  which is southwest of  $\tilde{C}$ ,  $d'(b) = d(b)$ ,
- for any  $b \in \mathcal{B}_w$  which is northeast of  $\tilde{C}$  and strictly southwest of  $C$ ,  $d'(b) = d(b) + 1$ .

The rest of the section is devoted to the proof of the above theorem, which mostly consists of (tedious) careful analysis of the  $r$ -worth of reverse paths starting from the four regions.

Let  $d, d', C, \tilde{C}$  be as in the theorem, except we do not assume  $C$  is not  $N$ -terminating.

**16.2.2. Alternative definition of backward numbering.** First we narrow the class of reverse paths that need to be considered. In the next result we show that the path does not need to cross  $C$  multiple times. In the second lemma we show that considering paths that bridge  $C$  is often unnecessary.

**Definition 16.20.** A path  $(b_0, \dots, b_l)$  for  $w$  is in *normal form* if for some  $0 \leq g \leq h \leq l+1$  we have

- $b_0, \dots, b_{g-1}$  all lie strictly to one side (southwest or northeast) of  $C$ ,
- $b_g, \dots, b_{h-1}$  are consecutive elements of  $C$ ,
- $b_h, \dots, b_l$  all lie strictly to one side (southwest or northeast) of  $C$ .

**Lemma 16.21.** Suppose we have a reverse path  $p = (b_0, \dots, b_k)$  for  $w$ . Then there exists a reverse path  $p' = (b'_0 = b_0, \dots, b'_l = b_k)$  in normal form of at most the same  $r$ -worth.

*Proof.* If the reverse path never intersects or bridges the channel, then we can take  $p' = p$ ,  $g = h = 0$ . Thus we only need to show that if  $p$  intersects or bridges the channel at two points, then the path obtained by following the channel between these points will have at

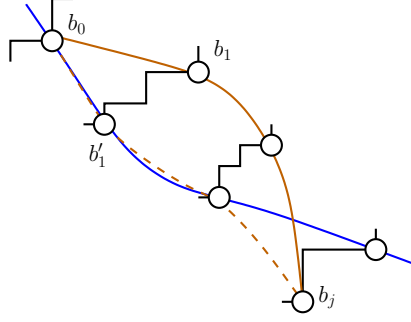


FIGURE 32. Getting a path into normal form.

most the same worth. There are four cases to consider depending on whether there are bridges or intersections at the two points. The cases are similar, so we only consider one in detail. Suppose that the reverse path first intersects the channel and then bridges the channel (see Figure 32, where the zig-zags considered are the ones from the induced numbering of  $\text{bk}_S(w)$ ).

Without loss of generality, we can consider  $b_0$  to be the point of intersection. Let  $j$  be such that  $b_j$  is the first ball after the bridge. For each  $1 \leq i < j$ , we can let our alternate path take the element of  $C$  in the same zig-zag as  $b_i$ . If there are zig-zags which  $p$  does not intersect, then including the elements of  $C$  will only decrease the  $r$ -worth of the alternate path as compared to the worth of  $p$ . Finally, it is clear that  $b_j$  is southeast of the entry of  $C$  in the zig-zag of  $b_{j-1}$ . The alternate reverse path is shown by the dashed line in the figure.  $\square$

**Lemma 16.22.** *Suppose we have a reverse path  $p = (b_0, \dots, b_k)$  for  $w$  in normal form; moreover suppose that  $p$  bridges  $C$ . Then there exists a reverse path  $p' = (b'_0 = b_0, \dots, b'_l)$  in normal form of the same  $r$ -worth which fords  $C$ .*

*Proof.* The zig-zags considered in the problem come from the induced backward numbering on  $\text{bk}_S(w)$ . Choose  $i$  such that  $b_i$  is on one side of  $C$  while  $b_{i+1}$  is on the other. Without loss of generality,  $b_i$  is southwest of  $C$  and  $b_{i+1}$  is northeast of  $C$ . Let  $b'_i$  (resp.  $b'_{i+1}$ ) be the ball of  $C$  on the zig-zag of  $b_i$  (resp.  $b_{i+1}$ ). Recall that  $m$  is our notation for the width of the Shi poset of  $w$ ; let  $b'_{i+2}, b'_{i+3}, \dots, b'_{i+m} = b'_i + (n, n)$  be consecutive elements of  $C$  after  $b'_{i+1}$ . Now the reverse path  $(b_0, \dots, b_i, b'_{i+1}, \dots, b'_{i+m}, b_{i+1} + (n, n), \dots, b_k + (n, n))$  fords  $C$  and has the same  $r$ -worth as  $p$ .  $\square$

**Lemma 16.23.** *Suppose  $b \in \mathcal{B}_w$  is on one side of  $C$  and  $\tilde{b} \in \mathcal{B}_w$  is on the other. If there is a reverse path from  $b$  to  $\tilde{b}$  of  $r$ -worth  $l$  then there exists a reverse path for  $w'$  from  $b$  to  $\tilde{b}$  of  $r$ -worth  $\leq l$ .*

*Similarly suppose  $b' \in \mathcal{B}_{w'}$  is on one side of  $C'$  and  $\tilde{b}' \in \mathcal{B}_w$  is on the other. If there is a reverse path from  $b'$  to  $\tilde{b}'$  of worth  $l$  then there exists a reverse path for  $w$  from  $b'$  to  $\tilde{b}'$  of worth  $\leq l$ .*

*Proof.* Suppose first that  $b$  is northeast of  $C$  and  $\tilde{b}$  is southwest of  $C$ . Suppose there exists a reverse path from  $b$  to  $\tilde{b}$  of  $r$ -worth  $l$ . Consider a path  $p = (b_0 = b, \dots, b_k = \tilde{b})$  in normal form with  $r$ -worth  $\leq l$ ; it necessarily fords or bridges  $C$ . Consider  $g$  and  $h$  such that  $b_g, \dots, b_{h-1}$



are the elements of the path on  $C$ . Let  $b'_g, \dots, b'_{h-1}$  be the balls of  $w'$  directly north of  $b_g, \dots, b_{h-1}$ . Now  $b_{g-1}$  is northwest of  $b'_g$  (it is clearly west of  $b'_g$  and it must be northeast of some element of  $C$ , so it is north of  $b_{g-1}$ ). So  $(b_0, \dots, b_{g-1}, b'_g, \dots, b'_{h-1}, b_h, \dots, b_k)$  is a path in  $w'$  of the same  $r$ -worth  $l$ .

If  $b$  is southwest of  $C$  and  $\tilde{b}$  is northeast of  $C$ , the argument is the same, but we need to take  $b'_g, \dots, b'_{h-1}$  directly east of  $b_g, \dots, b_{h-1}$ . The argument for the statement in the second paragraph is exactly the same.  $\square$

**16.2.3. Effect of shifting a channel on backward numbering.** In the next two lemmas, we describe the backward numbering on  $C$ .

**Lemma 16.24.** *Choose a ball  $b \in C$ . Let  $b' \in C'$  be the ball directly north of  $b$ , and let  $b'' \in C$  be the ball directly west of  $b'$ . Then either  $d'(b') = d(b)$  or  $d'(b') = d(b'')$ .*

*Proof.* We will show that  $d'(b') \leq d(b)$  and that  $d(b'') \leq d'(b')$ . Since  $d(b'') = d(b) - 1$ , this will imply the desired conclusion. We will only do the first inequality since the proof of the second one is exactly the same.

Consider a reverse path in normal form  $p = (b_0 = b, b_1, \dots, b_k)$  starting at  $b$  and having minimal  $r$ -worth. Either  $b_k \in C$  or not. For each  $i$  with  $b_i \in C$ , let  $b'_i$  be the ball of  $C'$  directly north of  $b_i$ . If  $b_k \in C$ , then  $(b'_0 = b', b'_1, \dots, b'_k)$  is a reverse path starting at  $b'$  and it has at most the  $r$ -worth of  $p$ . Now suppose  $b_k \notin C$ . Let  $h$  be such that  $b_h$  is the last ball of  $p$  on  $C$ . Then  $(b'_0 = b', b'_1, \dots, b'_h, b_{h+1}, \dots, b_k)$  is a reverse path starting at  $b'$  and it has at most the  $r$ -worth of  $p$ .  $\square$

*Remark 16.25.* The last ball on a reverse path of minimal  $r$ -worth is necessarily terminal.

**Lemma 16.26.** *Choose a ball  $b \in C$ . Let  $b' \in C'$  be the ball directly north of  $b$ , and let  $b'' \in C$  be the ball directly west of  $b'$ . The channel  $C$  is  $N$ -terminating if and only if  $d'(b') = d(b'')$ .*

*Proof.* Suppose that  $d'(b') = d(b'')$ . Consider a reverse path  $p' = (b'_0 = b', b'_1, \dots, b'_k)$  in normal form (with respect to the channel  $C'$ ) of minimal  $r$ -worth. For each  $i$  with  $b'_i \in C'$ , let  $b_i$  be the ball of  $C$  directly south of  $b'_i$ . There are three cases depending on whether  $b'_k$  is on  $C'$ , southwest of  $C'$  or northeast of  $C'$ .

Suppose  $b'_k$  is on  $C'$ . Consider the reverse path  $p = (b_0 = b, b_1, \dots, b_k)$ . Let  $d_0$  be the stream numbering (we don't need to make a distinction between  $w$  and  $w'$  since for balls present in both partial permutations the stream numbering is the same). Since  $b'_k$  is terminal, we have  $d_0(b'_k) = d'(b'_k) = d(b_{k-1})$ . So  $S^{(d_0(b'_k)+1)}$  is not northwest of  $b'_k$  and hence not northwest of  $b_{k-1}$ . The path  $p$  has the same number of steps as the path  $p'$ , so by assumption  $d_0(b_k) > d_0(b'_k)$ . Hence the element of the stream  $S^{(d_0(b_k)+1)}$  must be northwest of  $b_k$ . So  $S^{(d_0(b_k)+1)}$  is south of  $b_{k-1}$ , and hence  $b_{k-1}$  is  $N$ -terminal.

Suppose  $b'_k$  is southwest of  $C'$ . Let  $h$  be the index of the last ball  $b'_h$  of  $p'$  on  $C'$ . Then  $(b_0 = b, b_1, \dots, b_h, b'_{h+1}, \dots, b'_k)$  is a reverse path for  $w$  starting at  $b$  and having the same  $r$ -worth as  $p'$ . This is a contradiction.

Suppose  $b'_k$  is northeast of  $C'$ . Let  $h$  be the index of the last ball  $b'_h$  of  $p'$  on  $C'$ . Then  $p = (b_0 = b, b_1, \dots, b_{h-1}, b'_{h+1}, \dots, b'_k)$  is a reverse path for  $w$  starting at  $b$  and having  $r$ -worth one greater than than of  $p'$  (we were forced to skip a step since we have no guarantee that  $b'_{h+1}$  is south of  $b_h$ ). Thus  $p$  is a minimal  $r$ -worth reverse path from  $b$  and  $b'_k$  is terminal for  $w$  (and of course it was terminal for  $w'$ ). It cannot be  $N$ -terminal by the assumption that  $C$  is not  $N$ -terminating, so it must be  $W$ -terminal. But if it were  $W$ -terminal, then the element

of  $C'$  in its zig-zag would also be  $W$ -terminal. Thus we could have chosen  $p'$  to stay on  $C'$ , reducing the problem to a previous case. This proves the first half of the statement.

Now we prove that if  $C$  is  $N$ -terminating then  $d'(b') = d(b'')$ . Consider a minimal  $r$ -worth reverse path  $p = (b_0 = b'', b_1, \dots, b_k)$  from  $b''$  in normal form such that  $b_k$  is  $N$ -terminal. For each  $i$  with  $b_i \in C$ , let  $b_i'''$  be the ball directly east of  $b_i$ . We may assume that  $b_k$  is weakly northeast of  $C$  since otherwise there is a  $N$ -terminal ball on  $C$  and one can replace  $p$  by the reverse path along the channel. Moreover if  $b_k \in C$  then  $d'(b_k''') \leq d(b_k)$  and Lemma 16.24 finishes the proof. Finally, if  $b_k$  is strictly northeast of  $C$ , let  $i$  be the largest index such that  $b_i \in C$ . Then  $(b_0''' = b', b_1', \dots, b_i''', b_{i+1}, \dots, b_k)$  is a reverse path in  $w'$  starting at  $b'$  and having  $r$ -worth equal to  $d(b'')$ . Hence  $d'(b') \leq d(b'')$  and Lemma 16.24 again finishes the proof.  $\square$

Combining the lemma with its reflection in the main diagonal yields the following corollary.

**Corollary 16.27.** *The channel  $C$  is  $N$ -terminating if and only if  $C'$  is not  $W$ -terminating.*

In the next lemma we describe the backward numbering strictly northeast of  $C$ ; it turns out to be the same for  $w$  and  $w'$ .

**Lemma 16.28.** *Suppose  $C$  is not  $N$ -terminating. Then for any ball  $b \in \mathcal{B}_w$  strictly northeast of  $C$ ,  $b$  is also a ball of  $w'$  and  $d'(b) = d(b)$ .*

*Proof.* Let  $b \in \mathcal{B}_w$  be strictly northeast of  $C$ . We want to show that for any reverse path (in normal form) in  $w$  starting at  $b$ , there exists a reverse path in  $w'$  of at most the same  $r$ -worth, and vice versa. We omit the second part of the proof since it is nearly identical to the first one.

Consider a reverse path  $p$  (in normal form) starting at  $b$  in  $w$ . If the path does not intersect  $C$ , then the same path has the same  $r$ -worth in  $w'$ . Thus we now assume  $p$  intersects  $C$ .

Suppose  $p = (b_0 = b, b_1, \dots, b_k)$ . Let  $h$  be the first index such that  $b_h \in C$ . Let  $b_h'$  be the ball of  $C'$  directly north of  $b_h$ . Choose a path  $(b_h', \dots, b_l')$  of smallest  $r$ -worth in  $w'$ . Since  $d(b_h) = d'(b_h')$ , the  $r$ -worth of the reverse path  $(b_0, \dots, b_{h-1}, b_h', \dots, b_l')$  is at most the same as the  $r$ -worth of  $p$ .  $\square$

In the next three statements we describe the backward numbering strictly southwest of  $\tilde{C}$ ; it turns out to be the same for  $w$  and  $w'$ .

**Lemma 16.29.** *Suppose  $C$  is not  $N$ -terminating and  $b$  is some ball southwest of  $\tilde{C}$ . Then there exists a path of minimal  $r$ -worth such that every element of the path is southwest of  $\tilde{C}$ .*

*Proof.* Suppose  $b$  is a ball of  $w$  southwest of  $\tilde{C}$ . Consider the semi-bounded reverse walk starting at  $b$  and staying southwest of  $\tilde{C}$ . If the walk is forced to stop, then we are done since it must stop at a  $W$ -terminal element. Otherwise, it necessarily reaches some channel  $B$  of  $w$ ; let  $p = (b_0 = b, b_1, \dots, b_k)$  be the initial part of the walk such that  $b_k$  is the first element in  $B$ . Now  $b_k$  is southwest of  $C$  (in fact, southwest of  $\tilde{C}$ ), hence  $b_k$  is  $W$ -terminating. Consider a reverse path  $q$  (in normal form) starting at  $b_k$  and having minimal  $r$ -worth. The path  $q$  may be chosen to stay (weakly) southwest of  $B$ , since if it ended northeast of  $B$ , the ball of  $B$  in the same zig-zag as the ending would be  $W$ -terminal, hence one could just follow  $B$  to the terminal ball. The path formed by following  $p$  and then following  $q$  satisfies the desired properties.  $\square$

So in the above case a path of minimal worth in  $w$  is also a path in  $w'$ :

**Corollary 16.30.** *Suppose  $C$  is not  $N$ -terminating and  $b$  is some ball southwest of  $\tilde{C}$ . Then  $d'(b) \leq d(b)$ .*

**Lemma 16.31.** *Suppose  $C$  is not  $N$ -terminating and  $b$  is some ball southwest of  $\tilde{C}$ . Then  $d'(b) = d(b)$ .*

*Proof.* Consider a reverse path (in normal form)  $p'$  for  $w'$  which starts at  $b$ . We will show that there exists a reverse path  $p$  for  $w$  which starts at  $b$  and has at most the same  $r$ -worth as  $p'$ . This implies  $d(b) \leq d'(b)$ , and an application of the previous corollary will finish the proof.

If  $p'$  did not intersect  $C'$  then there is nothing to prove; hence we assume that it did. Let  $(b'_0 = b, b'_1, \dots, b'_k)$  be the initial segment of  $p'$  such that  $b'_k$  is the first element of the path on  $C'$  (hence  $b'_{k-1}$  is southwest of  $C'$ ). Let  $b_k$  be the element of  $C$  directly west of  $b'_k$ ; then  $d(b_k) < d'(b'_k)$ . Let  $(b_k, b_{k+1}, \dots, b_l)$  be a reverse path of minimal  $r$ -worth starting at  $b_k$ . Then the reverse path  $(b'_0 = b, b'_1, \dots, b'_{k-1}, b_k, b_{k+1}, \dots, b_l)$  is a reverse path in  $w$  of smaller  $r$ -worth than  $p'$ .  $\square$

In fact, the proof of the above lemma shows that a minimal  $r$ -worth path in  $w'$  must not intersect  $C'$ . Finally, in the next two results, we finish the description of the backward numbering of  $w'$ .

**Lemma 16.32.** *Consider a ball  $b \in \mathcal{B}_w$  which is northeast of  $\tilde{C}$  and strictly southwest of  $C$ . If  $d(b) = k$ , then  $b$  is east of the element  $c$  of  $\tilde{C}$  in the  $(k+1)$ -st zig-zag.*

*Proof.* It is clear that  $b$  is north of  $c$ . Suppose  $b$  is west of  $c$ . Let  $c'$  be the ball of  $\text{bk}_S(w)$  directly north of  $b$ ;  $c'$  is northeast of  $\tilde{C}$  and (strictly) southwest of  $C$ . Consider a walk  $(c_0 = c', c_1, \dots, c_l)$  from  $c'$  to  $\tilde{C}$ . Then the path  $(c, c_0, c_1, \dots, c_l)$  is a path from  $\tilde{C}$  to itself which visits consecutive zig-zags, stays southwest of  $C$ , and contains elements strictly northeast of  $\tilde{C}$ . This contradicts the fact that  $\tilde{C}$  was the northwest channel of  $\text{bk}_S(w)$  southwest of  $C$ .  $\square$

*Remark 16.33.* Notice an interesting consequence of the previous lemma. First, there cannot be  $W$ -terminal balls northeast of  $\tilde{C}$  and strictly southwest of  $C$ . If in every zig-zag there are balls in the region described, then of course there cannot be  $W$ -terminal balls northeast of  $C$ . Moreover, no reverse path which visits consecutive zig-zags can cross  $\tilde{C}$  in this case. So in this case  $C$  could not have been  $W$ -terminating. Thus if  $C$  is  $W$ -terminating then there exists a ball of  $C$  directly east of a ball of  $\tilde{C}$ .

**Lemma 16.34.** *Suppose  $C$  is not  $N$ -terminating. Consider a ball  $b \in \mathcal{B}_w$  which is northeast of  $\tilde{C}$  and strictly southwest of  $C$ . Then  $d'(b) = d(b) + 1$ .*

*Proof.* We want to show that  $d'(b) \leq d(b) + 1$  and  $d(b) \leq d'(b) - 1$ . In terms of reverse paths, we want to show that for every reverse path in  $w$  from  $b$  there exists a path in  $w'$  whose  $r$ -worth is at most one greater, and for every reverse path in  $w'$  from  $b$  there exists a path in  $w$  whose  $r$ -worth is at least one less. We only do the first of these since the second one is exactly the same. Consider a reverse path  $p = (b_0 = b, b_1, \dots, b_k)$  in  $w$  of minimal  $r$ -worth in normal form.

By the above remark,  $p$  intersects or bridges  $C$  at some point; by Lemma 16.22 we only need to consider the case when  $p$  intersects  $C$ . Suppose  $b_i$  is the first ball of  $p$  on  $C$ . Let  $b'_i$

be the ball of  $C'$  directly east of  $b_i$ . Let  $(b'_i, b'_{i+1}, \dots, b'_l)$  be a reverse path of minimal  $r$ -worth starting at  $b'_i$ . Then, since  $d(b_i) = d'(b'_i) - 1$  the path  $(b_0, \dots, b_{i-1}, b'_i, \dots, b'_l)$  has  $r$ -worth at most one greater than that of  $p$ .  $\square$

#### 16.2.4. Effect of moving a channel on the result of the backward step.

**Lemma 16.35.** *Let  $w$  be a partial permutation and  $S$  be a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Choose a non- $N$ -terminating river of  $w$  and let  $C$  be its northeast channel. Let  $\tilde{C}$  be the northeast channel of  $\text{bk}_S(w)$  which is southwest of  $C$ . Then  $\tilde{C}$  is the northeast channel in its river.*

*Proof.* This follows from an easy adaptation of Theorem 15.1 to the case of arbitrary channel numberings, noting that the indexing river here is northeast of  $C$ .  $\square$

**Proposition 16.36.** *Let  $w$  be a partial permutation and  $S$  be a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Choose a non- $N$ -terminating river of  $w$  and let  $C$  be its northeast channel. Let  $w' = w \langle 1 \rangle_C$ . Let  $C' = C \langle 1 \rangle$ . Let  $\tilde{C}$  be the northeast channel of  $\text{bk}_S(w)$  which is southwest of  $C$ . Then  $\text{bk}_S(w') = \text{bk}_S(w) \langle 1 \rangle_{\tilde{C}}$ .*

*Proof.* Notice a couple of consequences of Theorem 16.19. Suppose  $c$  and  $c'$  are two consecutive elements of  $\tilde{C}$ . Then all the elements on the zig-zag of  $c$  strictly between  $C$  and  $\tilde{C}$  lie east of  $c'$  (otherwise the numbering described in the theorem would not be monotone). Similarly, the ball of  $C'$  directly north of a ball  $b \in C$  must lie north of the element of  $C$  preceding  $b$ .

Now using Theorem 16.19 we can analyze the zig-zags for  $\text{bk}_S(w')$ , which can be seen in Figure 33. The zig-zags corresponding to  $\text{bk}_S(w)$  are shown in black. The zig-zags corresponding to  $\text{bk}_S(w')$  are shown in green. The black balls are balls of  $w$ , the green balls are balls of  $C' \subset \mathcal{B}_{w'}$ , and the magenta balls are balls of  $\tilde{C} \subset \mathcal{B}_{\text{bk}_S(w)}$ . As can be seen from the picture, northwest corners of the black zig-zags and green zig-zags almost always coincide, except the collection of black corners giving  $\tilde{C}$  gets replaced by a collection of green corners corresponding to a shift of  $\tilde{C}$  by 1. Hence the two partial permutations differ by a shift of  $\tilde{C}$ , as desired.  $\square$

We can complete our description of how the result of the backward step changes when we shift a river.

**Proposition 16.37.** *Let  $w$  be a partial permutation and  $S$  be a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Choose a  $W$ -terminating river of  $w$  and let  $C$  be its southwest channel. Let  $w' = w \langle -1 \rangle_C$ . Let  $C' = C \langle -1 \rangle$ . Let  $B$  be the northeast channel of  $\text{bk}_S(w)$  southwest of  $C$ ; let  $\tilde{C}$  be the southwest channel of the river of  $B$ . Then  $\text{bk}_S(w') = \text{bk}_S(w) \langle -1 \rangle_{\tilde{C}}$ .*

*Proof.* By Corollary 16.27 we know that  $C'$  is not  $N$ -terminating. Thus, given Proposition 16.36, we need to show that if  $B'$  is the northwest channel of  $\text{bk}_S(w')$  southwest of  $C'$ , then  $\tilde{C} = B' \langle 1 \rangle$ . We already know that  $B' \langle 1 \rangle$  is the southwest channel of its river and that  $\text{bk}_S(w) = (\text{bk}_S(w')) \langle 1 \rangle_{B'}$ . If the channel  $B$  did not intersect  $B' \langle 1 \rangle$  then we would have a contradiction with the definition of  $B'$ . So  $B'$  is part of the river of  $B$ , as desired.  $\square$

We can also understand what happens when we shift a channel multiple times.

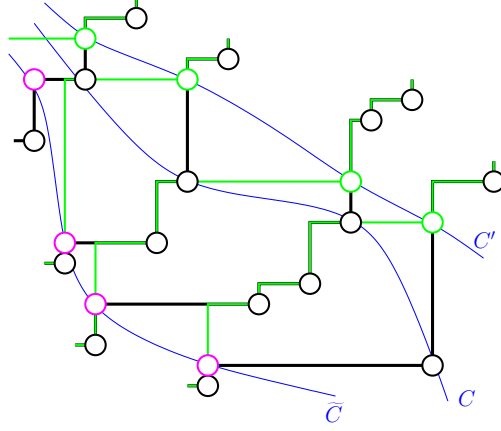


FIGURE 33. The zig-zags of  $\text{bk}_S(w)$  (in black) and of  $\text{bk}_S(w')$  (in green).

**Proposition 16.38.** *Suppose  $w$  is a partial permutation,  $S$  is a compatible stream whose flow is equal to the width of the Shi poset of  $w$ , and  $C$  is the northeast channel of a river which is not  $N$ -terminating. Let  $\tilde{C}$  be the northeast channel of  $\text{bk}_S(w)$  southwest of  $C$ . Let  $C' = C \langle 1 \rangle$ ,  $\tilde{C}' = \tilde{C} \langle 1 \rangle$ , and  $w' = w \langle 1 \rangle_C$ . Let  $C''$  be the northeast channel of the river of  $w'$  containing  $C'$ , and let  $\tilde{C}''$  be the northwest channel of the river of  $\text{bk}_S(w')$  containing  $\tilde{C}'$ . Let  $w'' = w' \langle 1 \rangle_{C''}$ .*

*Then*

$$\text{bk}_S(w'') = \text{bk}_S(w') \langle 1 \rangle_{\tilde{C}''}.$$

*Proof.* Suppose first that  $C'$  is also not  $N$ -terminating; let  $\tilde{C}'$  be the result of shift in  $\tilde{C}$  by 1. Suppose we shift again the river containing  $C'$ ; let  $C''$  be the northeast channel of that river. One only needs to prove that  $\tilde{C}'$  is part of the same river as the northeast channel  $\tilde{C}''$  of  $\text{bk}_S(w')$  southwest of  $C''$ .

Since the indexing river is located northeast of  $\tilde{C}''$ , it is sufficient to show that there exists a path from  $\tilde{C}''$  to  $\tilde{C}'$  which visits consecutive zig-zags. Now  $C'$  and  $C''$  are part of the same river, there exists a path from  $C''$  to itself which intersects  $C'$  such that the corresponding river numbering drops by 1 at each step. Since the backward numbering and the river numbering have the same period, it must be that the backward numbering also drops down by one at each step. Hence there is a path from  $C''$  to  $C'$  such that the backward numbering drops down by 1 at each step. Continue this path along  $C'$ . Then a (bounded or semi-bounded) walk leads from  $\tilde{C}''$  to  $\tilde{C}'$  and visits consecutive zig-zags.

Now suppose  $C'$  is  $N$ -terminating (and of course, still,  $W$ -terminating). Since  $C$  contains a ball directly east of  $\tilde{C}$ ,  $C'$  contains a ball directly east of a ball of  $\tilde{C}'$ . Now this ball must be  $N$ - and  $W$ -terminating, so  $\tilde{C}'$  is part of the indexing river. Now any ball directly north of a ball of  $C''$  is also part of the indexing river. Hence, using the reflection of Proposition 16.37 in the main diagonal, we see that the channel to be shifted to get  $\text{bk}_S(w'')$  is exactly the northwest channel of the indexing river, i.e.  $\tilde{C}''$ .  $\square$

### 16.3. Weyl symmetry.

**Lemma 16.39.** *Suppose  $w$  is a partial permutation,  $T$  is a compatible stream, and  $S$  is a stream compatible with  $w' := \text{bk}_T(w)$ . Moreover suppose that  $T$  is concurrent to  $S$  (in particular,  $T$  and  $S$  have the same flow). Then the indexing river of  $w'$  is both  $N$ -terminating and  $W$ -terminating with respect to  $S$ .*

*Proof.* Choose a proper numbering  $d_S$  of  $S$ ; let  $d_T$  be the backward numbering of  $T$  with respect to  $S$ . Let  $d$  be the induced numbering of  $w'$  from the backward step with respect to  $T$ . Let  $d'$  be the backward numbering of  $w'$  with respect to  $S$ . Since  $d$  is a monotone numbering such that for each  $i$ ,  $S^{(i)}$  is northwest of all balls labeled  $i$ , we know that for all  $b \in \mathcal{B}_{w'}$ ,  $d(b) \leq d'(b)$ .

Since  $T$  is concurrent to  $S$ , we can choose  $i$  such that  $T^{(i)}$  is north of  $S^{(i+1)}$ . Consider  $b \in \mathcal{B}_{w'}$  directly east of  $T^{(i)}$ . By definition of  $d'$ ,  $d'(b) \leq i = d(b)$ , so  $d'(b) = d(b)$ . Thus  $b$  is  $N$ -terminal with respect to  $S$ . Now choose a path  $(b_0 = b, b_1, \dots, b_k)$  such that  $b_k$  is on the indexing river of  $w'$  and  $d$  decreases by 1 at each step. Since  $d'$  must also decrease at each step, we have  $d(b_i) = d'(b_i)$  for all  $i$ . So the indexing river of  $w'$  is  $N$ -terminating. By reflecting the arguments in the main diagonal, we see that the indexing river of  $w'$  is  $W$ -terminating.  $\square$

**Lemma 16.40.** *Suppose  $w$  is a partial permutation and  $S$  is a compatible stream whose flow is equal to the width of the Shi poset of  $w$ . Suppose  $w$  has a river  $R$  which is both  $W$ -terminating and  $N$ -terminating. Let  $w' = w \langle -1 \rangle_R$  and  $w'' = w \langle 1 \rangle_R$ . Then  $\text{bk}_S(w')$  differs from  $\text{bk}_S(w)$  by shifting the indexing river by  $-1$ , and  $\text{bk}_S(w'')$  differs from  $\text{bk}_S(w)$  by shifting the indexing river by  $1$ .*

*Proof.* It is sufficient to prove the statement for  $w'$ ; the one for  $w''$  will follow by reflection in the main diagonal. Let  $C$  be the southwest channel of  $R$ . We only need to show that the channel  $\tilde{C}$  of  $\text{bk}_S(w)$  from Proposition 16.37 is part of the indexing river; this is true because the ball directly west of a ball of  $C$  is clearly part of the indexing river.  $\square$

We are now ready for the main theorem of this section.

**Theorem 16.41.** *Suppose  $w$  is a partial permutation,  $T = \mathbf{st}_r(A, B)$  is a compatible stream and  $S = \mathbf{st}_{r'}(A', B')$  is compatible with  $\text{bk}_T(w)$ . Let  $z$  be the unique integer such that  $\mathbf{st}_z(A, B)$  is concurrent to  $\mathbf{st}_0(A', B')$ . Let  $\tilde{T} = \mathbf{st}_{r'+z}(A, B)$  and  $\tilde{S} = \mathbf{st}_{r-z}(A', B')$ . Then  $\text{bk}_S(\text{bk}_T(w)) = \text{bk}_{\tilde{S}}(\text{bk}_{\tilde{T}}(w))$ .*

*Proof.* We know that  $\tilde{T}$  is concurrent to  $S$ . By the previous two lemmas, Theorem 16.8, and Remark 16.9 and Proposition 16.38, we know that for any  $k$ ,  $\text{bk}_S(\text{bk}_{\tilde{T} \langle k \rangle}(w)) = \text{bk}_{S \langle k \rangle}(\text{bk}_{\tilde{T}}(w))$ . In particular, for  $k = r - z - r'$  we have  $\tilde{T} \langle k \rangle = T$  and  $S \langle k \rangle = \tilde{S}$ , finishing the proof.  $\square$

This is the key to proving that there is a Weyl group action on the fibers of  $\Psi$ .

*Proof of Theorem 6.3.* Repeated application of the previous theorem shows that  $\Psi(P, Q, \rho) = \Psi(P, Q, \rho')$ . Now because  $\rho'$  is dominant,  $\Phi(\Psi(P, Q, \rho')) = (P, Q, \rho')$ .  $\square$

## REFERENCES

[Ari99] S. Ariki, *Robinson-Schensted correspondence and left cells*, arXiv:math/9910117, 1999.

- [BF01] T. Britz and S. Fomin, *Finite posets and Ferrers shapes*, Advances in Mathematics **158** (2001), no. 1, 86–127.
- [BV82] D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259** (1982), 153–199.
- [Ful97] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, 1997.
- [GK76] C. Greene and D. J. Kleitman, *The structure of  $S_n$ -perner  $k$ -families*, Journal of Combinatorial Theory, Series A **20** (1976), no. 1, 41–68.
- [GM88] A. M. Garsia and T. J. McLarnan, *Relations between Young’s natural and the Kazhdan-Lusztig representations of  $S_n$* , Advances in Mathematics **69** (1988), no. 1, 32–92.
- [Hon05] T. Honeywill, *Combinatorics and Algorithms Associated with the Theory of Kazhdan-Lusztig Cells*, Ph.D. thesis, University of Warwick, 2005.
- [KL89] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1989), 165–184.
- [Lus89] G. Lusztig, *Cells in affine Weyl groups, IV*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), no. 2, 297–328.
- [Shi86] J. Y. Shi, *Kazhdan-Lusztig cells of certain affine Weyl groups*, Lecture Notes in Mathematics, vol. 1179, Springer-Verlag, 1986.
- [Shi91] J.Y. Shi, *The generalized Robinson-Schensted algorithm on the affine Weyl group of type  $A_{n-1}$* , Journal of Algebra **139** (1991), no. 2, 364–394.
- [Vie77] G. Viennot, *Une forme géométrique de la correspondance de Robinson-Schensted*, Combinatoire et représentation du groupe symétrique (1977), 29–58.
- [Xi02] N. Xi, *The Based Ring of Two-Sided Cells of Affine Weyl Groups of Type  $A_{n-1}$* , Mem. Amer. Math. Soc., vol. 749, American Mathematical Soc., 2002.

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